Certified Rational Parametric Approximation of Real Algebraic Space Curves with Local Generic Position Method

Jin-San Cheng, Kai Jin

KLMM, Institute of Systems Science, AMSS, CAS, Beijing 100190, China

Daniel Lazard

Universite Pierre et Marie Curie, INRIA Paris-Rocquencourt Research Center

Abstract

In this paper, an algorithm to determine the topology of an algebraic space curve and to compute a certified G^1 rational parametric approximation of the algebraic space curve is given by extending the local generic position method for solving zero dimensional polynomial equation systems to the case of dimension one. By certified, we mean the approximation curve and the original curve have the same topology and their Hausdorff distance is smaller than a given precision. The main advantage of the algorithm, inherited from the local generic method, is that the topology computation and approximation for a space curve are directly reduced to the same tasks for two plane curves. In particular, the error bound of the approximation space curve is obtained from the error bounds of the approximation plane curves explicitly. We also analyze the complexity of computing the topology of an algebraic space curve. Nontrivial examples are used to show the effectivity of the method.

Key words: Real algebraic space curve, topology, complexity, rational approximation parametrization, local generic position

1. Introduction

Polynomial system solving is a main topic in symbolic computation. When considering a zero-dimensional system, we usually isolate its (real) roots or get its approximating (real) roots. For a positive dimensional system, we usually want to obtain its (real) dimension, its topology, or its approximate representation. Algebraic space curves have many applications in computer aided geometric design, computer aided design, and geometric modeling. For example, the algebraic space curves defined by two quadrics are widely used in geometric

The corresponding author: Jin-San Cheng, Fax: +86-10-6263-0706.
Email:jcheng@amss.ac.cn(Jin-San Cheng), jinkaijl@163.com(Kai Jin), daniel.lazard@lip6.fr(Daniel Lazard).

modeling. One can have an exact parametrization for these algebraic space curves. However, exact parametrization representations for general algebraic space curves do not exist. So the use of approximate techniques is unavoidable for parametrization of algebraic space curves. One usually approximates an algebraic space curve with piecewise rational curves under a given precision. Moreover, it is important to ensure that the approximation curves preserve the topology of the original algebraic space curve. We call the approximation **certified** (at precision ϵ) if it has the same topology as the original one and the Hausdorff distance between the curve and its approximation is upper-bounded (by ϵ).

There are several difficulties for approximate parametrization of algebraic space curves. The first one is to preserve the topology of the algebraic space curve. In fact, there already exist nice works of computing the topology of algebraic space curves (Alcazar and Sendra (2005); Daouda et. al. (2008); El Kahoui (2008); Gatellier et. al. (2005); Owen and Rockwood (1987)). Most of them require the curve to be in a generic position. For the space curves which are not in a generic position, one needs to take a coordinate transformation on the space curves such that the new space curves are in a generic position. Thus some geometric information of the original space curves is lost. Some non-singular critical points of the new space curves may not correspond to the non-singular critical points of the original space curves. One needs additional computation to get these points in the original coordinate system. Subdivision method can preserve the topology of the curve in a theoretical sense. But it is rather difficult to reach the required bound in practice currently (Burr et. al. (2008); Liang et. al. (2008)). Even if one gets the topology of the given curves, the approximation curves may have different topology as the original curves when two or more curve segments are very close(see Fig. 2). We will provide a new method to compute the topology of algebraic space curves. The second difficulty is the error control of the approximation curve. Some error functions are reliable but it is not easy to compute in practice, for example (Chuang and Hoffmann (1989)). We need to find a reliable and efficient method to control the error during the approximation. The third one is the continuity of the approximation curve. We usually require the approximation to be C^1(or G^1)-continuous or higher in practice. Doing so, we need to compute the tangent directions of the algebraic space curve at some points. It is not a trivial task especially when the component considered is *non-reduced*. Its tangent direction can not be decided by the normal directions $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, $(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$ of the two surfaces $f = 0$ and $g = 0$ at the given point. In the non-reduced case, two normal directions are parallel or at least one of them does not exist at the point.

There exist nice works about approximating of algebraic space curves. Exact parametrization of the intersection of algebraic surfaces is obtained in (Abhyankar and Bajaj (1989); Berry (1997); Bizzarri and Lavicka (2011); Dupont et. al. (2008a,b,c); Gao and Chou (1992); Hoeij (1997); Schicho (1992); Sendra and Winkler (1991); Tu et. al. (2009); Wang et. al. (2003)). Of course, it is topology preserving. The approximation of the intersection of generic algebraic surfaces with numeric method is also considered (Bajaj et. al (1988); Bajaj and Xu (1994); Hartmann (2000); Jüttler and Chalmovianský (2007); Krishnan and Manocha (1997); Patrikalakis (1993); Pratt and Geisow (1986)). Usually, numeric method cannot guarantee the topology of the original algebraic space curve.

In (Béla and Jüttler (2000)), the authors considered approximating of the regular algebraic space curves with circular arcs by numeric method combining with subdivision method. It works well for low degree algebraic space curves. In (Gao and Li (2004)), the authors presented an algorithm to approximate an irreducible space curves under a given precision. It is based on the fact that there exists a birational map between the projection curve $\mathbb{C}$ for some direction and the irreducible algebraic space curve. But an irreducible decomposition of a given two polynomials system is not an easy task. When we decomposing a reducible space curve into irreducible ones, we need to check whether two or more irreducible space curves have intersection or not. Other types of intersection problem of surfaces can be found in (Patrikalakis (1993); Pratt and Geisow (1986)).

In (Bizzarri and Lavicka (2012)), the authors present an algorithm to approximate an algebraic space curve, defined by $f = g = 0$, in the generic position with Ferguson's cubic $\mathbf{p}(t) = (x(t), y(t), z(t)), t \in [0, 1]$ and by minimizing an integral to control the error. They compute the topology of the space curve at first, so it is topology preserving. They do not check whether the approximation curve exactly preserves the topology of the original space curve. The method works well for regular space curves.

In (Rueda et. al. (2012)), the authors consider the irreducible algebraic space curve in generic position such that its projection is birational. They use a genus 0 plane algebraic curve to approximate the projection plane curve under a given precision if it exists. Thus they have a rational approximation space curve for the original space curve. The method is not topology preserving.

In this paper, we present a new algorithm to compute the topology as well as a certified G^1 rational parametric approximation for algebraic space curves, which solves the three difficulties mentioned above nicely. The algorithm is certified in the sense that the approximation curve and the original curve have the same topology and their Hausdorff distance is smaller than a given precision. The algorithm works for algebraic space curves which need not to be regular nor in generic positions. The key idea is to extend the local generic position method (Cheng et. al. (2012, 2009)) for zero-dimensional polynomial systems to one-dimensional algebraic space curves. The algorithm consists of three major steps.

Firstly, the space curve $\mathbb{S}$, which is the intersection of $f(x, y, z) = 0$ and $g(x, y, z) = 0$, is projected to the xy-plane as a plane curve $\mathbb{C}_1$ and $\mathbb{C}_1$ is approximated piecewisely with functions of the form $(x, w_1(x)), x \in [a, b]$.

Secondly, we find a number $s > 0$ such that

1) under the shear transformation $\varphi : (x, y, z) \to (x, y + s\,z, z)$, $\varphi(f)$ and $\varphi(g)$ are in a generic position in the sense that there is a one to one correspondence between the curve segments of $\mathbb{S}$ and that of their projection curve $\mathbb{C}_2$ to the xy-plane. The plane curve $\mathbb{C}_2$ is also approximated piecewisely with functions of the form $(x, w_2(x)), x \in [a, b]$.

2) we choose s such that $\mathbb{C}_2$ is in a **local generic position** to $\mathbb{C}_1$ in the following sense.

- The plane curves $\mathbb{C}_1$ and $\mathbb{C}_2$ can be divided into segments such that each segment of $\mathbb{C}_2$ corresponds to a segment of $\mathbb{C}_1$.
- Let $w_1(x), w_2(x), x \in [a, b]$ be the approximations for a segment C_1 of $\mathbb{C}_1$ and the corresponding curve segment C_2 of $\mathbb{C}_2$ with precisions ϵ_1 and ϵ_2 respectively. Then the space curve segment S corresponding to C_2 can be approximated by $(x, w_1(x), \frac{w_2(x)-w_1(x)}{s})$ with precision $\sqrt{s^2\epsilon_1^2 + (\epsilon_1 + \epsilon_2)^2}/s$.

In other words, if $\mathbb{C}_2$ is in a local generic position to $\mathbb{C}_1$, then each segment of the space curve can be represented as a linear combination of corresponding segments of $\mathbb{C}_1$ and $\mathbb{C}_2$. As a consequence, a certified parametrization for the space curve can be computed from that of $\mathbb{C}_1$ and $\mathbb{C}_2$ directly. To find out the correspondence between the curve segments of $\mathbb{C}_1, \mathbb{C}_2$ is the most important step of the method and thus it is the main contribution of the paper. We obtain the topology of the space curve from the topology of $\mathbb{C}_1, \mathbb{C}_2$ and the correspondence of $\mathbb{C}_1$ and $\mathbb{C}_2$. This two steps are the **main contribution** of this paper, from which we can transform a 3-dimensional problem to a 2-dimensional problem.

Finally, we show that a plane curve can be approximated such that the piecewise approximation curve for the space curve has G^1 continuity and has one of the following forms: $(x, a\,x+b+\frac{c}{d\,x+1}, a_1\,x+b_1+\frac{c_1}{d_1\,x+1}+\frac{c_2}{d_2\,x+1})$, $(a\,y+b+\frac{c}{d\,y+1}, y, a_1\,y+b_1+\frac{c_1}{d_1\,y+1}+\frac{c_2}{d_2\,y+1})$.

The paper is organized as below. In the next section, we will consider the certified approximation of plane algebraic curve under a given precision. In Section 3, we will show the theory and algorithm for topology determination and certified approximation of algebraic space curves. We also analyze the complexity of computing the topology of an algebraic space curve. In Section 4, we will show some examples to illustrate the effectivity of our method. We draw a conclusion in the last section.

2. Approximate parametrization of plane algebraic curves

Given a plane algebraic curve defined by a square free polynomial $f \in \mathbb{Q}[x, y]$, our aim is to give a piecewise C^1-continuous approximation of $\mathbb{C} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$ in a given box $B = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$ such that each piece of the approximation curve has the form $(x, w(x))$ such that $\|f(x, w(x))\| < \epsilon << 1$ and the approximation error is bounded by a given precision $\epsilon > 0$, where $\mathbb{Q}, \mathbb{R}$ are the fields of rational numbers and real numbers, respectively. The whole approximation curve has the same topology as $\mathbb{C}$.

2.1. Notations and definitions

In this subsection, we will introduce some notations.

A point $P = (x_0, y_0)$ is said to be a **singular point** on $\mathbb{C} : f(x, y) = 0$ if $f(x_0, y_0) = f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, where $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}$. A non-singular point is called a **regular point**. An x**-critical** (A y**-critical**) point $P = (x_0, y_0)$ is a point satisfying $f(x_0, y_0) = f_y(x_0, y_0) = 0$ ($f(x_0, y_0) = f_x(x_0, y_0) = 0$). So a singular point is both x-critical and y-critical points. The **inflexion points** or **flexes** of $\mathbb{C}$ are its non-singular points satisfying its Hession equation $H(f) = 0$ (see (Walker (1978))).

A **regular curve segment** C of $\mathbb{C}$ is a connected part of $\mathbb{C}$ with two endpoints $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ ($x_0 \neq x_1$, both P_0, P_1 are bounded) and there are no x-critical points, y-critical points, flex on C except for P_0, P_1. Since the x-critical point and y-critical point appear only as endpoints of C, C is monotonous. And any flex on C is an endpoint. So C is convex. Let Δ be the triangle defined by P_0, P_1 and their tangent lines. Thus C is inside Δ. An endpoint of a regular curve segment is called a **vertical tangent point**, **VT** point for short, if the regular curve segment has a vertical tangent line at this endpoint.

A parametric curve is said to be C^1**-continuous** (G^1**-continuous**) if the curves are joined and the first derivatives are continuous (the curves also share a common tangent direction at the join point).

2.2. Curve segmentation of a real plane algebraic curve

In this subsection, we will show how to divide a plane curve inside a box B, denoted as $\mathbb{C}_B$, into regular curve segments with the form $[P_0(x_0, y_0), P_1(x_1, y_1), T_0(1, k_0), T_1(1, k_1)]$, where P_0, P_1 are endpoints and T_0, T_1 are tangent directions at the endpoints.

We will follow the steps below.

At first, compute the topology of $\mathbb{C}_B$. There are many related works to solve this problem, such as (Alberti et. al. (2008); Arnon and McCallum (1988); Berberich et al. (2012); Cheng et. al. (2010); Eigenwilling and Kerber (2008); Gonzalez-Vega and Necula (2002); Hong

(1996); Sakkalis (1991); Seidel and Wolpert (2005)). Some methods work well, but they need a coordinate system transformation. We prefer the methods which do not take a coordinate system transformation such as (Berberich et al. (2012); Cheng et. al. (2010); Hong (1996)). The main steps is as below.

(1) Compute $p(x) = \texttt{Res}_y(f, \frac{\partial f}{\partial y})$ and isolate its real zeros.
(2) For each real root α_i of $p = 0$, compute the roots $\beta_{i,j}$ of $f(\alpha_i, y) = 0$.
(3) Compute the numbers of left and right real branches of $f = 0$ originating from $(\alpha_i, \beta_{i,j})$.
(4) Construct topological graph from the points $(\alpha_i, \beta_{i,j})$.

Second, compute all the flexes, x-critical and y-critical points of $\mathbb{C}_B$.

Third, we split the plane curve into regular curve segments at $x(y)$-critical points or flexes of $\mathbb{C}_B$.

Finally, we represent the tangent direction of any non-VT point as $(1, k), k \in \mathbb{R} \setminus \{+\infty, -\infty\}$. The tangent direction of a VT point is defined to be $(1, \infty)$.

Tangent direction computation of singularities. In (Gao and Li (2004)), there is an algorithm to compute the tangent directions of an algebraic plane curve at a singularity. We can also compute them approximately as below, which is easy to implement. Let $P(\alpha, \beta)$ be a singularity of a planar algebraic curve $f(x, y) = 0$ and $C : (x, \tilde{y}(x)), x \in [\alpha, \gamma]$ a regular curve segment originating from right side (left side is similar) of P. Then the tangent direction of C at P is $(1, t) = (1, \lim_{x \to \alpha^+} \tilde{y}(x))$. Let $[a, b]$ be the isolating interval of $\alpha (\neq a, \neq b)$. We can use the tangent direction of some regular point on C close to P to replace the tangent direction of C at P. For instance, $(1, \frac{\partial \tilde{y}(b)}{\partial x}) = (1, -\frac{f_x}{f_y}(b, \tilde{y}(b)))$ can be regarded as the tangent direction of C at P, as shown in Fig. 1. When C has a vertical tangent direction, $t = \infty$. If $|\frac{\partial \tilde{y}(b)}{\partial x}| > N$, for example, $N = 100$, we can regard the regular curve segment have a vertical tangent direction. Note that we can not guarantee that the branch has a vertical tangent direction here. But the approximation is still C^1-continuous. The choose of N depends on experiments. The second method is easy to implement with less computation. We need not to compute the tangent directions of the branches at a singularity which is time-consuming.

If we cannot distinguish the tangent directions of two groups of regular curve segments, we can refine $[a, b]$ to a narrower one and recompute the tangent directions again until we can distinguish them or they are less than some given bounded value τ such that $|k - k'| < \tau$, where k, k' are tangent directions.

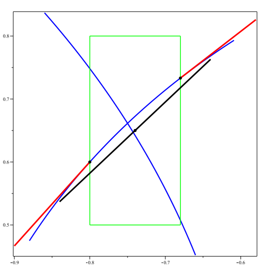

Fig. 1. Approximate the tangent direction of regular curve segments at a singularity

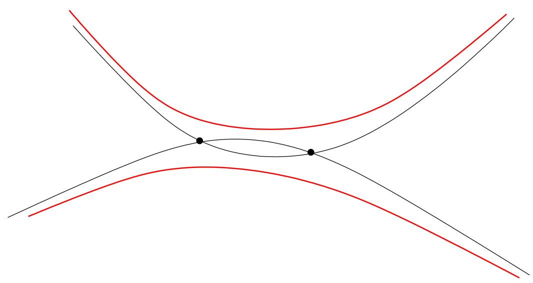

Fig. 2. The approximation curves change the topology of original ones

2.3. Approximation of a regular curve segment

We will give an approximate parametrization of real plane algebraic curves. Though Gao and Li (Gao and Li (2004)) have obtained a rational quadratic approximation of real plane algebraic curves with B-splines, and other related works such as (Bajaj and Xu (1997); Gahleitner et al. (2002); Pérez-Diaz et. al. (2010); Sendra and Winkler (1991)), we need to derive a piecewise approximation curve as $(x, w(x))$ of real plane algebraic curves in order to approximately parameterize real space algebraic curves in a different way. Let C be the regular curve segment defined by two points $P_0(x_0, y_0), P_1(x_1, y_1)$. We divide the approximation problem into two cases.

C **not containing a VT point.** Given a regular curve segment C, its two endpoints $P_0(x_0, y_0), P_1(x_1, y_1)$ does not contain a VT point. And the tangent directions of the regular curve segment at P_0, P_1 are $(1, k_0), (1, k_1)$, respectively. We will construct an explicit rational quadratic function $Y_1(x)$ to approximate C such that $Y_1(x_i) = y_i, Y_1'(x_i) = k_i, i = 0, 1$, where $Y_1'(x) = \frac{\partial Y_1}{\partial x}(x)$.

Assuming that

$$Y_1(x) = \frac{a\,x^2 + b\,x + c}{d\,x + 1} \qquad (1)$$

and $x_0 = 0, x_1 = 1$, we have

$$Y_1(0) = y_0, Y_1(1) = y_1, Y_1'(0) = k_0, Y_1'(1) = k_1.$$

Solving a, b, c, d from the equations above, we have

$$a = \frac{-2\,y_0\,y_1 + {y_0}^2 - k_0\,k_1 + {y_1}^2}{-y_1 + k_1 + y_0},$$
$$b = -\frac{-2\,y_0\,y_1 + y_0\,k_1 + 2\,{y_0}^2 + k_0\,y_1 - k_0\,k_1}{-y_1 + k_1 + y_0},$$
$$c = y_0,$$
$$d = -\frac{-2\,y_1 + k_0 + k_1 + 2\,y_0}{-y_1 + k_1 + y_0}.$$

From the representation, we need to require

$$-y_1 + k_1 + y_0 \neq 0, \qquad (2)$$

and $d\,x + 1$ has no roots in $[0, 1]$, that is, $d > -1$, from which we can derive that

$$(-y_1 + k_0 + y_0)\,(-y_1 + k_1 + y_0) < 0. \qquad (3)$$

From the mean value theorem, we know that $y_1 - y_0 = k_x$, where k_x is the tangent direction of some $x \in (0, 1)$. Since C is monotonous, so k_x is some value between k_0 and k_1. Thus conditions (2) and (3) are satisfied directly. We can easily transform the interval $[0, 1]$ to $[x_0, x_1]$ by setting $x = \frac{X - x_0}{x_1 - x_0}$, where $x \in [0, 1]$ when $X \in [x_0, x_1]$.

Furthermore, when $d \neq 0$, that is to say $-2\,y_1 + k_0 + k_1 + 2\,y_0 \neq 0$. Then expression (1) can be transformed into

$$Y_1(x) = \tilde{a}\,x + \tilde{b} + \frac{\tilde{c}}{\tilde{d}\,x + 1}. \qquad (4)$$

Though equation (4) is equivalent to equation (1) when $d \neq 0$ essentially, it has a simpler form and can reduce computation when evaluation. When $d = 0$, equation (1) is a polynomial of degree two. And we have simple expressions for parameters a, b, c, that is $a = \frac{k_1 - k_0}{2}$, $b = k_0$, $c = y_0$.

C **containing a VT point.** When a given regular curve segment contains a VT point, it means the tangent line at P_0 or P_1 is a vertical line $x - x_0 = 0$ or $x - x_1 = 0$. In this case, the method above does not work. But we can use part of an ellipse or a hyperbola $h(x, y) = \frac{(x-x_o)^2}{a^2} \pm \frac{(y-y_o)^2}{b^2} - 1 = 0 (a > 0, b > 0)$ to derive an approximate parametrization of a real plane algebraic curve. Note that a regular curve segment containing a VT point has four cases which exactly correspond to the four parts of an ellipse or a hyperbola: the vertical line is $x - x_0 = 0$ or $x - x_1 = 0$ and $y \geq y_o$ or $y \leq y_o$ (see Fig. 3). We consider the case that C has a vertical tangent line at $P_0(x_0, y_0)$ and C monotonously increases from P_0 to P_1 (case A in Fig. 3) to illustrate our method. And we assume that the tangent direction at P_1 is $(1, k_1)$ and the approximating curve has the form $Y_2(x) = y_o + \frac{b\sqrt{|a^2-(x-x_o)^2|}}{a}$. When we get x_o, y_o, a, b, we get the approximating curve.

Note that we have $x_o = x_0 + a$ (or $x_o = x_0 - a$), $y_o = y_0$ from the property of the ellipse (or the hyperbola). So we have

$$Y_2(x) = y_0 + \frac{b\sqrt{|a^2 - (x - x_0 \pm a)^2|}}{a}. \tag{5}$$

And $Y_2(x_1) = y_1, Y_2'(x_1) = -\frac{\partial h}{\partial x} / \frac{\partial h}{\partial y} = k_1$. Solving it, we have

$$a = \pm \frac{(x_1 - x_0)(x_0 k_1 + y_1 - y_0 - x_1 k_1)}{y_1 - y_0 - 2 x_1 k_1 + 2 x_0 k_1},$$
$$b = (x_0 k_1 + y_1 - y_0 - x_1 k_1) \sqrt{\pm \frac{y_1 - y_0}{y_1 - y_0 - 2 x_1 k_1 + 2 x_0 k_1}}.$$

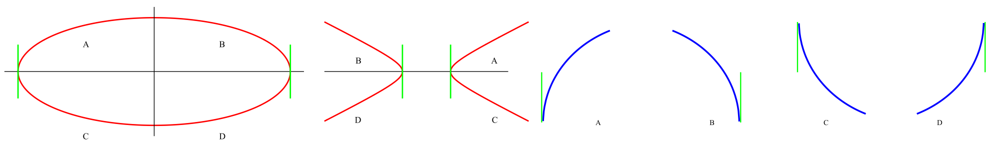

Fig. 3. Approximating a segment with part of an ellipse or a hyperbola. Given a segment with a VT point marked with $K(= A, B, C, D)$, one needs to use part of an ellipse or a hyperbola with the same mark to approximate it. The green vertical lines are tangent directions.

From the representation, we can find that a, b are well defined if $k_1 < \frac{y_1 - y_0}{2(x_1 - x_0)}$ for an ellipse or $k_1 > \frac{y_1 - y_0}{2(x_1 - x_0)}$ for a hyperbola. So we can choose (x_1, y_1) on the regular curve segment such that $k_1 \neq \frac{y_1 - y_0}{2(x_1 - x_0)}$. Then we can use part of an ellipse or a hyperbola to approximate the regular curve segments with VT points. The other three cases can be solved in a similar way.

For the two kinds of approximation curves above which are part of a hyperbola or an ellipse, say $\tilde{C}$, it is not difficult to find that it has the following properties:

(1) $\tilde{C}$ is monotonous in x.

(2) $\tilde{C}$ is convexity.

So we can find that $\tilde{C}$ is inside the triangle defined by C.

Topology preserving approximation. After we get the approximation regular curve segments, we need to check whether the approximation curve changes the topology of the original curve. Even if we get the correct topology of the given algebraic planar curve, the approximation curve may have a different topology as the original curve, especially when two regular curve segments are very close, for example, see Fig. 2. So we need to ensure that our numeric approximation curve has the same topology as the original one. We need only ensure that any two approximation curves, say $C_1, C_2 : (x, p(x)), (x, q(x)), x \in [a, b]$, are disjoint. If $p(x) - q(x) = 0$ has no real roots in (a, b), then the two approximation regular curve segments are disjoint. There are two kinds of approximation curves, say $Y_1(x), Y_2(x)$. We can check whether C_1, C_2 are disjoint or not by their monotonicity and convexities with only their endpoints. Otherwise, we need to consider the following three cases:

(1) Two approximation curves are both rational ones as $Y_1(x)$. Then $T(x) = p(x) - q(x)$ can be simplified into a cubic univariate polynomial. It is easy to check whether it contains a real roots in (a, b) by its coefficients.

(2) One is as $Y_1(x)$ and the other is as $Y_2(x)$. Then $T(x) = p(x) - q(x)$ can be simplified into a quartic univariate polynomial. Thus we have its roots with a formula in its coefficients. If the real roots are not inside (a, b), they are disjoint. Otherwise, we evaluate the real root in Y_1, Y_2 to check whether they are equal or not. It is not difficult to check whether it contains a real roots in (a, b) and two curve segments are disjoint or not.

(3) Both approximation curves are as $Y_2(x)$. They both are parts of quadric algebraic curves. Considering the intersection of two quadric algebraic curves, we can judge whether C_1, C_2 are disjoint or not. We can also get a quartic univariate polynomial. We can solve the problem similarly as above.

When $Y_1(x), Y_2(x)$ are not disjoint, we can subdivide the original curve into several ones and approximate them until the approximation curves are disjoint. Doing so as above, our approximation is exactly topology preserving.

2.4. Error control of the approximation

We will show the error control of the plane approximation curve in this subsection. In geometry, the approximation error should be defined as the following Hausdorff distance between the segment S and its approximation S_a,

$$e(S, S_a) = dis(S, S_a) = \max_{P \in S} \min_{P' \in S_a} d(P, P'). \tag{6}$$

However such a distance is difficult to compute. As an implement, the distance from an approximation parametric curve $P(t) = (x(t), y(t)), 0 \le t \le 1$ to the implicit defined curve $\mathbb{C} : f(x, y) = 0$ is taken in the following form, which is called the error function (Chuang and Hoffmann (1989)),

$$e(t) = \frac{f(x(t), y(t))}{\sqrt{(f_x(x(t), y(t))^2 + f_y(x(t), y(t))^2}}. \tag{7}$$

The approximation error between $P(t)$ and $\mathbb{C}$ is set as an optimization problem

$$e(P(t), \mathbb{C}) = \max_{0 \le t \le 1} (e(t)).$$

Let $C : (x, \tilde{y}(x)), x \in [x_0, x_1]$ be the regular curve segment and $\overline{C} := (x, Y(x)), x \in [x_0, x_1]$ its approximation curve. It is not difficult to find that the following bound is an upper bound of the Hausdorff distance between the segment C and its approximation curve $\overline{C}$ from (6):

$$\max_{x \in [x_0, x_1]} |Y(x) - \tilde{y}(x)|. \tag{8}$$

We use Newton-Ralphson method to obtain $\tilde{y}(x_i^0)$ at some point $x_i^0 \in [x_0, x_1]$ in practice and $Y(x_i^0)$ is the start point. If we fail to get a point with Newton-Ralphson method or the point satisfying $|Y(x_i^0) - \tilde{y}(x_i^0)| \geq \delta$, we can divide the regular curve segment into two ones. The approximation error is bounded by $\max_i\{|Y(x_i^0) - \tilde{y}(x_i^0)|\}$. In practice, for a given error ϵ, we sample x_i^0 as $x_i^0 = x_0 + i/n\,(x_1 - x_0), 0 \leq i \leq n$, for a proper positive integer of n related to $h_1 = \frac{x_1 - x_0}{\epsilon}$ and the slope $h_2 = \|\frac{Y_1(x_1) - Y_0(x_0)}{x_1 - x_0}\|$ in our experiments. When $h_1(h_2)$ increases, n increases.

2.5. Subdivision of curve segments

In order to control the error under a given precision, we need to divide the regular curve segment into two regular curve segments recursively until the error requirement satisfied. In this subsection, we will show how to subdivide the curve segment and that the subdivision is convergent. For any regular curve segment $C : (x, \tilde{y}(x)), x \in [a, b]$, we denote the endpoints as $P_0(x_0, y_0), P_1(x_1, y_1)$ and the tangent directions as $(1, k_i), i = 0, 1$. One can subdivide the regular curve segment into two ones, for example, $C_1 : x \in [x_0, (x_0 + x_1)/2], C_2 : x \in [(x_0 + x_1)/2, x_1]$, if the precision is not satisfied. For the approximation curve of our method, we will prove that it can achieve any given precision.

THEOREM 1. Let P_0, P_1 be the endpoints of a regular curve segment C and Δ the triangle related to the regular curve segment as defined before. The Hausdorff distance between C and its approximation curve(s) tends to zero if we subdivide C into two regular curve segments recursively.

Proof. Let $P((x_0 + x_1)/2, \bar{y})$ be a point on C. Denote the triangles formed by $P_1, P(P, P_2)$ and the tangent directions of C at these points as $\Delta_1(\Delta_2)$. Let the lengths of the line segments $\overline{P_1P}, \overline{PP_2}$ be L_1^1, L_2^1 and the heights of the triangles Δ_1, Δ_2 corresponding to the edges $\overline{P_1P}, \overline{PP_2}$ are H_1^1, H_2^1, as shown in Fig. 4. Subdividing the regular curve segments recursively in a similar way, denoting the length of the edges and heights as L_j^i, H_j^i of the triangles, we have the sum of the areas of these triangles are

$$A = \sum_j (L_j^i\, H_j^i/2) < \frac{1}{2} \sum_j L_j^i \max_j H_j^i.$$

When we subdivide the curve segment(s), $\sum_j L_j^i$ trends to the length of the arc of C and H_j^i trends to zero. Thus A trends to zero. Since both the approximation curve(s) and the given curve segment(s) are inside the triangle(s), we can find that we prove the theorem. □

Computing the subdivision points. There are two ways to find the subdivision points on a given regular curve segment $C : (x, \tilde{y}(x)), x \in [a, b]$. Since we get the topology of the plane projection curve, we know the order of the given regular curve segment among all the regular curve segments of the projection curve $h(x, y) = 0$ when x changes from a to b. That is, we can find the point on C for a fixed x coordinate, say $x_0 \in (a, b)$. It is the real root with the same order of $h(x_0, y) = 0$ in a fixed interval (or $(-\infty, +\infty)$).

Another way is a local method. We can trace the regular curve segment to find the point on C with given x coordinate since the regular curve segments are monotonous and convex. From a start point P of the regular curve segment, compute the tangent line of the regular curve segment at $P = (x_P, y_P)$, find a point $Q = (x_Q, y_Q)$ on the tangent line such that the step length $|x_P - x_Q|$ is bounded and the line segment $\overline{PQ}$ has no intersection with the projection curve. Then from Q, we can compute a point $P_1 = (x_Q, y_{P_1})$ on the regular curve segment. Doing so recursively, we can find the desired point, as shown in Fig. 5.

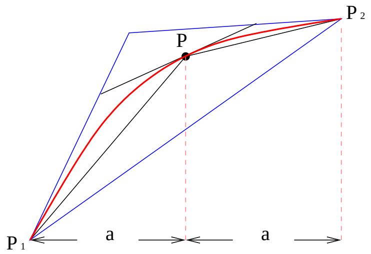

Fig. 4. Bisecting a regular curve segment

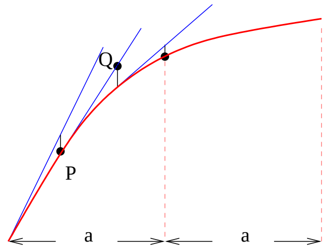

Fig. 5. Finding a subdivision point by tracing

With the preparation above, we have the following algorithm to approximate a plane algebraic curve.

Algorithm 1. The inputs are $\mathbb{C} : f(x, y) = 0$, a bounding box B and an error bound $\delta > 0$. The outputs are parametric curves $\mathbb{C}_1 := \{B_i(x) = (x, y_i(x)), a_i \leq x \leq b_i, (i = 1, ..., N)\}$, such that they give a C^1-continuous and topology preserving approximation to $\mathbb{C}_B$ with $e(\mathbb{C}_1, \mathbb{C}) < \delta$.

(1) Regular curve segmentation of $\mathbb{C}_B$ as in Section 2.2.
(2) Regular curve segment approximation as in Section 2.3 with error control of the approximation as in Section 2.4 and Section 2.5.

The correctness of the algorithm is clear from the analysis above. The termination of the algorithm is guaranteed by Theorem 1.

3. Certified approximate parametrization of algebraic space curves

In this section, we will consider certified approximate parametrization of algebraic s-pace curves. We recall the main steps mentioned in the introduction below. We compute a projection curve $\mathbb{C}_1 = \pi_z(\mathbb{S})$ of the space curve $\mathbb{S} = f \wedge g$, the intersection of $f = 0$ and $g = 0$, and piecewise approximate parameterize it in the form $(x, w_1(x))$ at first, where $\pi_z : (x, y, z) \to (x, y)$, $f, g \in \mathbb{Z}[x, y, z]$ and $\mathbb{Z}$ is the set of integers. Compute a rational number s such that $\varphi(\mathbb{S})$ is in a (local) generic position, where $\varphi : (x, y, z) \to (x, y + s\,z, z)$. Then we compute a projection curve $\mathbb{C}_2 = \pi_z(\varphi(\mathbb{S}))$ of the space curve $\varphi(\mathbb{S})$ and piecewise approximate parameterize it in the form $(x, w_2(x))$. For each curve segment C_2 of $\mathbb{C}_2$, we need to find out one curve segment $C_1 \subset \mathbb{C}_1$ such that $C_1 = \pi_z \circ \varphi^{-1} \circ \pi_z^{-1}(C_2)$. Then we can recover the corresponding space curve segment $S = \varphi^{-1} \circ \pi_z^{-1}(C_2)$ from the parametric representations of C_1 and C_2. And reparameterize it into rational one if needed. The approximation error is considered during we approximating the plane curves.

Since we require the plane projection curve can be parameterized in form $(x, w(x))$, we have the following assumption for the algebraic space curve.

- For any $x_0 \in \mathbb{R}$, $f(x_0, y, z) = g(x_0, y, z) = 0$ has a finite number of solutions; and
- the leading coefficients of f, g w.r.t. z have no common factors only in x.

In fact, most of the problems we considered satisfy the condition. Note that we can exchange x, y, z freely. And another coordinate system transformation $(x, y, z) \to (x+s\,z, y, z)$ can help us to find out the missing regular curve segments in the first transformation, even when the algebraic space curve contains vertical lines. Thus we can remove the assumptions with the method mentioned here. But we still assume the two assumptions holds in this section.

Since we have discussed how to compute the topology and approximation of plane curves in Section 2, we need only to compute s, to compute the correspondence between curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$. We will show it below.

3.1. Computing s

In this subsection, we will compute a good s such that $\varphi(\mathbb{S})$ is in a generic position and with some local property. In order to reduce the 3D approximation of space curves into 2D approximation of plane curves, we need the concept of local generic position. We recall the related definitions for zero-dimensional bivariate polynomial system (Cheng et. al. (2009)). Let $\mathfrak{C}$ be the field of complex numbers. Let $f, g \in \mathbb{Q}[x, y]$. We say two plane curves defined

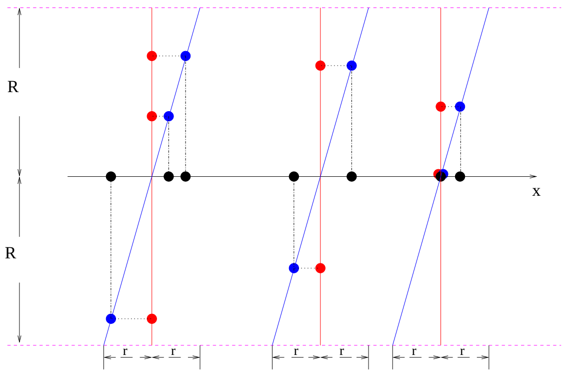

Fig. 6. LGP method

by two polynomials f, g such that $\gcd(f, g) = 1$ are in a **generic position** w.r.t. y if

1) The leading coefficients of f and g w.r.t. y have no common factors.

2) Let h be the resultant of f and g w.r.t. y. For any $\alpha \in \mathfrak{C}$ such that $h(\alpha) = 0$, $f(\alpha, y), g(\alpha, y)$ have only one common zero in $\mathfrak{C}$.

Then we will introduce the technique of local generic position (LGP for short) method.

Given $f, g \in \mathbb{Q}[x, y]$, their intersection (the red points in Fig. 6) not necessarily to be in generic position, we can take a coordinate transformation $\phi : (x, y) \to (x + s\,y, y), s \in \mathbb{Q}$ such that

- The intersection (the blue points in Fig. 6) of $\phi(f), \phi(g)$ is in a generic position w.r.t. x.
- Let $\bar{h}, h$ be the resultants of $\phi(f), \phi(g)$ and f, g w.r.t. y, respectively and $\pi_y : (x, y) \to (x)$. Each root α of $h(x) = 0$ has a neighborhood interval H_α such that $H_\alpha \cap H_\beta = \emptyset$ for roots $\beta \neq \alpha$ of $h = 0$. And any root (γ, η) of $f = g = 0$ which has a same x-coordinate γ, is mapped to $\gamma' = \pi_y \circ \varphi(\gamma, \eta) = \gamma + s\,\eta \in H_\gamma$ (γ' is as the black points in Fig. 6), where $h(\gamma) = 0, \bar{h}(\gamma') = 0$. Thus we can recover $\eta = \frac{\gamma' - \gamma}{s}$.

The method has two nice properties: 1) The 2D solving problem is transformed into a 1D solving problem. 2) The error control of the solutions is easier.

We say that two algebraic surfaces defined by $f, g \in \mathbb{Q}[x, y, z]$ such that $\gcd(f, g) = 1$ are in a **generic position** w.r.t. z if

1) The leading coefficients of f and g w.r.t. z have no common factors.

2) Let h be the square free part of the resultant of f and g w.r.t. z. There are only a finite number of zeros $(\alpha, \beta) \in \mathfrak{C}^2$ such that $h(\alpha, \beta) = 0$, (α, β) is not a y-critical point of $h = 0$, and $f(\alpha, \beta, z), g(\alpha, \beta, z)$ have more than one distinct common zeros in $\mathfrak{C}$.

The definition is similar to the definition of pseudo-generic position in (Daouda et. al. (2008)). In Theorem 4 of (Daouda et. al. (2008)), the authors also provide a method to check whether two given surfaces are in a pseudo-generic position or not, which can also be used to check whether an algebraic space curve is in a generic position or not.

We will use the technique of local generic position to compute s. Denote the real roots of $\texttt{Res}_y(h, h_y) = 0$ and the x-coordinates of the flexes and x-critical points of $h = 0$ as $\alpha_1, \alpha_3, \ldots, \alpha_{2t-1}$, where $h_y = \frac{\partial h}{\partial y}$. All these information can be obtained from Section 2. Find two rational numbers less than α_1 and larger than α_{2t-1}, denoted as α_0, α_{2t} respectively. For any two adjacent real roots $\alpha_{2i-1}, \alpha_{2i+1}$ of $\texttt{Res}_y(h, h_y) = 0$, we can find a rational number, say α_{2i}. Then we obtain a sequence $\alpha_i (i = 0, \ldots, 2t)$. Assume that the real roots of $h(\alpha_i, y) = 0$ are $\beta_{i,j} (j = 0, \ldots, t_i)$ which are listed in increasing order. We can find out that $(\alpha_i, \beta_{i,j})$ divide the plane curve $h = 0$ in the region $[\alpha_0, \alpha_{2t}] \times \mathbb{R}$ into regular curve segments (see definition in Section 2.1) and each curve segment has at least one regular point. Let

$$\begin{aligned} R &= \max_{0 \le i \le 2t, 0 \le j \le t_i} \mathrm{RB_z}(f(\alpha_i, \beta_{i,j}, z)), \\ r &= \min_{0 \le i \le 2t} \{R, \min_{0 \le j \le t_i - 1} (\beta_{i,j+1} - \beta_{i,j})\}, \end{aligned} \tag{9}$$

where $\beta_{i,-1} = -\infty$, $\mathrm{RB_z}(f(\alpha_i, \beta_{i,j}, z))$ is the root bound of $f(\alpha_i, \beta_{i,j}, z)$ in z, f can be replaced by g. We can choose only odd i to compute R, r for convenience. And $f(\alpha_i, \beta_{i,j}, z)$ can be replaced by $\gcd(f(\alpha_i, \beta_{i,j}, z), g(\alpha_i, \beta_{i,j}, z))$ for a better s. Then s can be chosen as a rational number satisfying:

$$0 < s < \frac{r}{2R}, s \in \mathbb{Q}. \tag{10}$$

After we choose s as above, the bivariate polynomial system $\{\phi(f(\alpha_i, y, z)), \phi(g(\alpha_i, y, z))\}$ has the property of local generic position for $i = 1, \ldots, 2t$.

Since it is probability 1 to obtain such an s that $\varphi(f(\alpha_i, y, z)), \varphi(g(\alpha_i, y, z))$ are in a generic position with the assumptions, it is probability 1 that $\varphi(f(\alpha_i, y, z)), \varphi(g(\alpha_i, y, z))$ are in a local generic position for all $\alpha_i (i = 0, \ldots, 2t)$ under condition (9). And we can ensure this by checking whether $\varphi(f) \wedge \varphi(g)$ is in a generic position.

When s is fixed, φ is fixed. We can simply call $\varphi(f \wedge g)$ is in a **local generic position** if $\varphi(f \wedge g)$ is in a generic position and s satisfying (9),(10).

3.2. Correspondence between $\mathbb{C}_1$ *and* $\mathbb{C}_2$

In this subsection, we will show how to determine the correspondence between the regular curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$. The main issue is to ensure that $\varphi(f \wedge g)$ is in a generic position and to find corresponding points on the regular curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$, respectively.

The following lemma give us a guide to find the correspondence between curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$.

Lemma 2. Let S be a space curve segment of $f \wedge g$, $C_1 = \pi_z(S)$ is a regular curve segment and with parametric representation $(x, w_1(x)), x \in [a, b]$ and, C_2 be a regular curve segment of $\pi_z(\varphi(f \wedge g))$ with parametric representation $(x, w_2(x)), x \in [a, b]$ and $\varphi(f \wedge g)$ is in a generic position. If $P = (x_0, w_1(x_0), z_0)$ is a point on S and

$$\pi_z(\varphi(P)) = (x_0, y_1(x_0) + s\, z_0) = (x_0, w_2(x_0)),$$

that is, $\pi_z(\varphi(P))$ is on C_2, then

$$S := \varphi^{-1} \circ \pi_z^{-1}(C_2), \text{ and } S := (x, w_1(x), \frac{w_2(x) - w_1(x)}{s}).$$

Proof. Since $\varphi(f \wedge g)$ is in a generic position, π_z^{-1} exists. C_1, C_2 are regular curve segments. P is a point on S and $\pi_z(\varphi(P))$ is on C_2, so $\varphi(S) = \pi_z^{-1}(C_2)$. Thus $S = \varphi^{-1} \circ \pi_z^{-1}(C_2)$ and $S = (x, w_1(x), \frac{w_2(x)-w_1(x)}{s})$. We can find that C_1 corresponds to C_2 since $\pi_z(P) \in C_1$. □

From the lemma, we require $\varphi(\mathbb{S})$ to be in a generic position and to find corresponding points on curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$. We can recover the space curve segments from the two plane projection curves by figuring out the correspondence between curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$. The key step is to find out the corresponding points on the curve segments of $\mathbb{C}_1$ and $\mathbb{C}_2$, respectively.

Let $C_1 \subset \mathbb{C}_1 : h(x, y) = 0$ be a regular curve segment with the form $(x, w_1(x)), x \in [\alpha, \alpha']$, where α, α' can be α_i, α_{i+1} for some i, where α_i is defined as in (9). Let $C_2 \subset \mathbb{C}_2$ be the curve segments of $\mathbb{C}_2$ corresponding to C_1 inside $[\alpha, \alpha'] \times \mathbb{R}$ (to be decided). But C_2 may contain non-singular critical points, flex or singular points. We call these points **key points** for simplicity. That is, C_2 may not be regular curve segments. We will discuss all these cases and show how to find out the correspondence between (part of) C_1 and (part of) C_2.

Case 1: C_2 contains no singular points. If C_2 contains no critical points nor flex, it is a regular curve segments. Otherwise, we can split C_2 into several regular curve segments at these nonsingular key points and parameterize them into form $(x, w_2(x)), x \in [a, b]$. And C_1 can also be split into several corresponding regular curve segments with the same x-domain as regular curve segments of C_2. Thus we can regard both $C_i : (x, w_i(x)), x \in [a, b], i = 1, 2$ as regular curve segments below.

Case 1a: regular endpoints of regular curve segments are correspondent. Let $P_1(t, y_1) \in C_1$ and $P_2(t, y_2) \in C_2$ be two endpoints for $t = a$ or $t = b$. If P_2 is a regular point of $\mathbb{C}_2$, P_2 is uniquely corresponding to a point on $\varphi(\mathbb{S})$ since $\varphi(\mathbb{S})$ is in a generic position. Furthermore, if P_1 is a regular point of $\mathbb{C}_1$, $P_1 = \pi_z(\varphi^{-1} \circ \pi_z^{-1}(P_2))$ and $t = \alpha_i$ or α_{i+1}, that is, y_2 is in a local region $(y_1 - r/2, y_1 + r/2)$ of y_1 for r as in (9). Thus we can find that C_1 corresponds to C_2 by Lemma 2. Most correspondence relationships can be determined in this way, for example, $\widetilde{BN}$ and $\widetilde{B_1N_1}$ in Fig. 7. N_1 corresponds to N and both are regular. Of course, if we know two regular endpoints of C_1, C_2 are correspondent, C_1, C_2 are correspondent.

Case 1b: regular curve segments correspondence can not be decided by its endpoints. We can choose a rational number between the x-coordinates of two endpoints of C_1 or C_2, say x_0, such that x_0 is not a zero of $\texttt{Res}_y(h, h_y)$ nor a zero of $\texttt{Res}_y(\bar{h}, \bar{h}_y)$. Solving $h(x_0, y) = 0$ and $\bar{h}(x_0, y) = 0$, we can get the y-coordinate of the point on C_1 by the order of the real root in all the real roots list (Note that we know the order when we computing the topology of $\mathbb{C}_1$), and the y-coordinates (denoted all as L) of the points on the curve segments of $\mathbb{C}_2$ on the fibre $x = x_0$. Lifting the point on C_1, say $P \in [x_0, x_0] \times [\gamma_1, \gamma_2]$, by interval method, that is, constructing a univariate interval polynomial and isolating the real roots of their bounding polynomials to get the roots of the interval polynomial (see detail in (Cheng et. al. (2009))), we get the real point candidates, presented by intervals, on $f \wedge g$ (say $[x_0, x_0] \times [\gamma_1, \gamma_2] \times [c_i, d_i], i = 1, \dots, k, k \geq 0$, denoted all as T). Refine, if needed, the isolating intervals in T such that $[\gamma_1, \gamma_2] + s[c_i, d_i]$ contains at most one element of L for each $i = 1, \dots, k$. For the elements in L, if they contained in $[\gamma_1, \gamma_2] + s[c_i, d_i]$ for some i, the corresponding curve segment of the elements in L corresponds to C_1 by the property of local generic position method. For example, we can decide the correspondences for the two regular curve segments defined by H, K with the method.

Case 2: C_2 contains singular points. When there exists a singular point(s) on $C_2 \subset \mathbb{C}_2$, the singular point(s) may correspond to a true singular point(s). It happens when $f \wedge g$ is not in a generic position. A regular curve segment of $\mathbb{C}_1$ may be not delineable (see the definition in McCallum's paper (McCallum (1985)) or other related papers), that is, there may exist a point on the regular curve segment (not an endpoint) corresponding to a singularity of $f \wedge g$. Two or more space curve segments of $f \wedge g$ intersect on a cylinder surface defined by some factor(s) of $h = 0$. So this kind of singularities of $f \wedge g$ may not correspond to singularities of $h = 0$. Since $\varphi(f \wedge g)$ is in a generic position, the singularity corresponds to a singularity of $\bar{h} = 0$. If two or more left (right) branches of a singularity of $\bar{h} = 0$ correspond to a same regular curve segment of $h = 0$, we can judge that it is a true singularity of $f \wedge g$, see the point D in Fig. 7 for example. The regular curve segments $\widetilde{A_1DL_2}, \widetilde{A_2DL_1}, \widetilde{A_3EFL_3}$ belonging to $\bar{h} = 0$ all correspond to $\widetilde{AL}$ since $A_1, A_2, A_3(L_1, L_2, L_3)$ are in a neighborhood of $A(L)$. Thus our method can recover the missing true singular point(s). The following shows how to determine the correspondence for the curve segments with a singular point(s).

Case 2a: the curve segments of C_2 have different tangent lines at the singular point(s). The left and right branches with the same tangent line are contained in the same curve segment by the continuity of the curve whatever the singularity related to a true singularity of $f \wedge g$ or not. Note that the tangent directions of the branches are decided when we compute the parametric representation of $\mathbb{C}_2$. Then we can decide the correspondence between C_1 and C_2 by their endpoints.

Case 2b: two or more curve segments of C_2 have the same tangent line at the singular point(s). If there exist two or more curve branches having same tangent lines at a singular point of $\bar{h} = 0$ in $(\alpha, \alpha') \times \mathbb{R}$, see point G in Fig. 7 for example, we call it **tangent false singularity**. If C_2 contains only one tangent false singularity in (α, α'), we can still find out the correspondence of C_2 and its corresponding regular curve segment C_1 following the methods in case 1 and case 2a. For G in Fig. 7, we know the endpoints of $\widetilde{R_3GU_3}$ are in the fixed neighborhood of R and U, so $\widetilde{R_3G}, \widetilde{GU_3}$ correspond to $\widetilde{RU}$. But if C_2 contains two or more tangent false singularities in (α, α'), we can not determine the correspondence of the part(s) of C_2 between these tangent false singularities and C_1 (or other regular curve segment of $h = 0$). As shown in Fig. 7, H, K are two tangent false singularities and we do not know the correspondence of the two regular curve segments between them directly. We can use the technique in case 1b to find the correspondence between (parts of) C_1 and (parts of) C_2.

Based on the analysis above, we get an algorithm to obtain the correspondence between the two projection curves $h = 0$ and $\bar{h} = 0$. Thus we get the topology of the algebraic space curve.

Some remarks for the method:

(1) If computing an algebraic space curve inside a box $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we need to decide the intersection between the space curve and the boundaries of the box. We can use two planes $z - a_3 = 0, z - b_3 = 0$ to intersect f, g and project the

intersection into xy-plane, respectively. Replace h by the product of these projections and h. Use two lines $y - a_2 = 0, y - b_2 = 0$ to intersect $h = 0$, we get some of the projections of the boundaries of the curve inside B, see Example 3 for detail.

(2) In the computation in practice, $(\alpha_i, \beta_{i,j})$ are represented by isolating intervals. The corresponding method to get r, R can be found in (Cheng et. al. (2009)).

If we approximate the two plane projection curves with the forms (1), (5), we can get the piecewise approximation parametric space curves of $f \wedge g$ from the correspondence of the regular curve segments in the two plane curves. They have the form:

$$(x, w_1(x)(w_2(x) - w_1(x))/s), x \in [a, b] \tag{11}$$

where $w_1(x), w_2(x)$ are as Forms (1), (5).

We need also check whether our approximation space curve changes the topology of original space curve or not. Since the plane approximation curve does not change the topology of the plane projection curve, we need only to check whether two approximation space regular curve segments having the same y coordinate are disjoint or not. We assume that the two approximation space regular curve segments are

$$C_1 : (x, w(x), \frac{w_2(x) - w(x)}{s}), C_2 : (x, w(x), \frac{w_1(x) - w(x)}{s}), x \in [a, b].$$

They are disjoint if $\frac{w_2(x)-w(x)}{s} - \frac{w_1(x)-w(x)}{s} = 0$ has no real roots in (a, b). We use the method as in Section 2.3.

3.3. Complexity of the algorithm to compute the topology of the space curve

In this subsection, we will discuss the complexity of the method above for computing the topology of a space curve. In this paper, $\mathcal{O}$ means the complexity and $\tilde{\mathcal{O}}$ indicates that we omit logarithmic factors. We use $\mathcal{O}_B$ to present bit complexity and $\tilde{\mathcal{O}}_B$ to present bit complexity ignoring logarithmic factors.

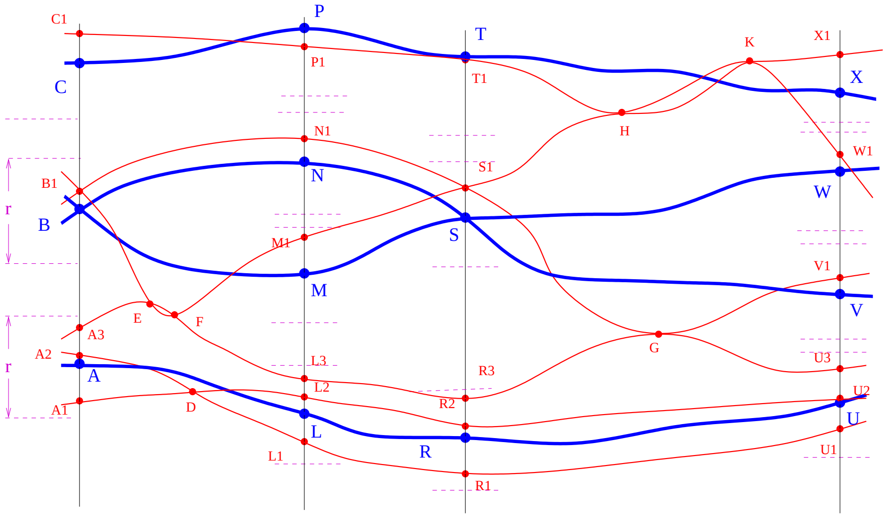

Fig. 7. The correspondence between the projection curves $\mathbb{C}_1 : h = 0$(thick) and $\mathbb{C}_2 : \bar{h} = 0$. Q_i corresponds to Q for $Q = A, B, C, L, M, N, P, R, S, T, U, V, W, S, i = 1, 2,$ or 3.

Lemma 3 (Basu et. al (2006); Mignotte (1992); Yap (2000)). Let $f(x)$ be a polynomial in $\mathbb{Z}[x]$ and $\deg_x(f) \leq d$, $\mathcal{L}(f) \leq \tau$. Then the root bound of $f = 0$ is $\mathcal{O}(\tau)$ and the separation bound of f is $\tilde{\mathcal{O}}(d\tau)$. The latter provides a bound on the bitsize of the endpoints of the isolating intervals.

Lemma 4 (Kerber and Sagraloff (2012)). Let $f \in Z[x, y]$ with $\deg(f) \leq d$ and $\mathcal{L}(f) = \tau$. We can compute the topology of $f = 0$ in $\tilde{\mathcal{O}}_B(d^9\tau + d^{10})$.

Lemma 5 (Sagraloff (2012)). For a square-free polynomial f with integer coefficients of modulus less than 2^τ, we can compute isolating intervals (for all real roots of f) of width less than 2^{-L} using no more than $\tilde{\mathcal{O}}_B(n^3\tau + n^2L)$ bit operations.

Lemma 6 (Diochnos et al (2009)). Let $f, g \in (Z[y_1, \ldots, y_k])[x]$ with $\deg_x(f) = p \geq q = \deg_x(g)$, $\deg_{y_i}(f) \leq p$ and $\deg_{y_i}(g) \leq q$, $\mathcal{L}(f) = \tau \geq \sigma = \mathcal{L}(g)$. We can compute $\mathtt{res}_x(f, g)$ in $\tilde{\mathcal{O}}_B(q(p + q)^{k+1}p^k\tau)$. And $\deg_{y_i}(\mathtt{res}_x(f, g)) \leq 2pq$, and the bitsize of coefficients for the resultant is $\tilde{\mathcal{O}}(p\sigma + q\tau)$.

Lemma 7. Let $f, g \in \mathbb{Z}[x, y, z]$ and $\deg(f), \deg(g) \leq d; \mathcal{L}(f), \mathcal{L}(g) = \tau$. Then the bitsize of s (see Equation (10) for definition) is bounded by $\tilde{O}(d^{11}\tau)$. And the bit complexity of computing s is $\tilde{\mathcal{O}}_B(d^{18}\tau)$.

Proof. Let h be the square free part of $\mathtt{Res}_z(f, g)$, $p(x) = \mathtt{Res}_y(h, \frac{\partial h}{\partial y})$ and $q(y) = \mathtt{Res}_x(p, h)$. Then $\deg(h) = \mathcal{O}(d^2)$, $\deg(p) = \mathcal{O}(d^4)$, $\deg(q) = \mathcal{O}(d^6)$ and $\mathcal{L}(h) = d\tau$, $\mathcal{L}(p) = d^3\tau$, $\mathcal{L}(q) = d^5\tau$ by Lemma 6. The bit complexities of computing h, p, q are $\tilde{\mathcal{O}}_B(d^4\tau)$, $\tilde{\mathcal{O}}_B(d^9\tau)$ and $\tilde{\mathcal{O}}_B(d^{17}\tau)$ respectively by Lemma 6. So the separation bound of $q(y)$ is $\tilde{\mathcal{O}}(d^{11}\tau)$ by Lemma 3. To get R, we can compute $v = \mathtt{Res}_y(f, h, y)$ in $\tilde{\mathcal{O}}(d^{12}\tau)$ and $\deg(v) = \mathcal{O}(d^3), \mathcal{L}(v) = \mathcal{O}(d^2\tau)$. And then to compute $e(z) = \mathtt{Res}_x(v, p)$ in $\tilde{\mathcal{O}}(d^{18}\tau)$ and $\deg(e) = \mathcal{O}(d^7), \mathcal{L}(v) = \mathcal{O}(d^6\tau)$. So $\mathcal{L}(R) = \mathcal{O}(d^6\tau)$ and $\mathcal{L}(r) = \tilde{\mathcal{O}}(d^{11}\tau)$. From Equation (10), we can find that $\mathcal{L}(s) = \mathcal{L}(r) = \tilde{\mathcal{O}}(d^{11}\tau)$. So the bit complexity of computing s is $\tilde{\mathcal{O}}_B(d^{18}\tau)$. □

Lemma 7 gives a worst complexity for s. But usually, s is not so bad.

Theorem 8. Let $f, g \in \mathbb{Z}[x, y, z]$, $\gcd(f, g) = 1$ and $\deg(f), \deg(g) \leq d; \mathcal{L}(f), \mathcal{L}(g) = \tau$. The bit complexity of computing the topology of the algebraic space curve defined by f, g is $\tilde{O}_B(d^{37}\tau)$.

Proof. The main steps to compute the topology of the space curve are computing $\mathbb{C}_1 : h(x, y) = 0$ and its topology, computing s, computing $\mathbb{C}_2 : \bar{h}(x, y) = 0$ and its topology, computing the correspondence relationship of $\mathbb{C}_1$ and $\mathbb{C}_2$.

From Lemma 6 and Lemma 4, the complexities of computing $\mathbb{C}_1 : h(x, y) = 0$ and its topology are $\tilde{\mathcal{O}}_B(d^4\tau)$ and $\tilde{\mathcal{O}}_B(d^{19}\tau + d^{20})$, respectively.

From Lemma 7, we can find that the bitsizes of the coefficients of $\varphi(f), \varphi(g)$ are bounded by $\tilde{\mathcal{O}}(d^{12}\tau)$, where $\varphi : (x, y, z) \to (x, y + s\,z, z)$. Their degrees are still bounded by d. So the bit complexity to compute $\bar{h}$ is $\tilde{\mathcal{O}}_B(d^{16}\tau)$ and $\deg(\bar{h}) = \mathcal{O}(d^2), \mathcal{L}(\bar{h}) = \tilde{\mathcal{O}}(d^{13}\tau)$ by Lemma 6. Thus the bit complexity to compute the topology of $\mathbb{C}_2$ is $\tilde{\mathcal{O}}_B(d^{31}\tau)$ from Lemma 4.

For the correspondence between the points of $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfying the local properties, the bit complexity to determine their correspondence can be ignored. We consider the correspondence between the curve segments which can not be determined by its endpoints. We need to choose an x_0 on the curve segments such that both $\mathbb{C}_1$ and $\mathbb{C}_2$ have no critical points on the fiber $x = x_0$, and to get real roots on $\{f(x_0, y, z), g(x_0, y, z)\}$ as shown in Case 1b to determine the correspondence. We need to choose at most $\mathcal{O}(d^4)$ different x_0. The bitsize of x_0 is bounded by the separation bound of the univariate polynomial $\mathtt{Res}_y(\bar{h}, \frac{\partial \bar{h}}{\partial y})\,\mathtt{Res}_y(h, \frac{\partial h}{\partial y})$. The degree of the polynomial is bounded by $\mathcal{O}(d^8)$ and the bitsize of the coefficients is bounded by $\tilde{\mathcal{O}}(d^{15}\tau)$. So $\mathcal{L}(x_0) = \tilde{\mathcal{O}}(d^{23}\tau)$. The bit complexity to get x_0 is bounded by $\mathcal{O}(d^4)^2\,\tilde{\mathcal{O}}(d^{23}\tau) = \tilde{\mathcal{O}}(d^{31}\tau)$ by Lemma 5. Solving $h(x_0, y) = 0$, we get the isolating intervals for the real roots. Their bitsizes are $\tilde{\mathcal{O}}(d^{27}\tau)$ by Lemma 3 since $\deg(h(x_0, y)) = \mathcal{O}(d^2)$ and $\mathcal{L}(h(x_0, y)) = \tilde{\mathcal{O}}(d^{25}\tau)$. There are at most $\mathcal{O}(d^2)$ different real roots. For some y_0, we can construct the interval polynomial by y_0's isolating interval for $f(x_0, y_0, z)$. Its degree is bounded by d and $\mathcal{L}(f(x_0, y_0, z)) = \tilde{\mathcal{O}}(d^{28}\tau)$. The bitzies of isolating intervals of the z-coordinates are $\tilde{\mathcal{O}}(d^{29}\tau)$. The bit complexity to get its real roots is $\tilde{\mathcal{O}}_B(d^{31}\tau)$ by Lemma 5. The bit complexity to get all the real roots for the given x_0, y_0 are bounded by $\mathcal{O}(d^4)\,\mathcal{O}(d^2)\,\tilde{\mathcal{O}}_B(d^{31}\tau) = \tilde{\mathcal{O}}_B(d^{37}\tau)$. Checking whether two curve segments are correspondent or not, we need to compute the multiplication of s and the isolating intervals of the real roots of $f(x_0, y_0, z) = 0$. Its complexity is $\tilde{\mathcal{O}}(d^{11}\tau) + \tilde{\mathcal{O}}(d^{29}\tau) = \tilde{\mathcal{O}}(d^{29}\tau)$. Checking all the roots is bounded by $\mathcal{O}(d^4)\,\mathcal{O}(d^2)\,\mathcal{O}(d)\,\tilde{\mathcal{O}}_B(d^{29}\tau) = \tilde{\mathcal{O}}_B(d^{36}\tau)$. Thus the worst case bit complexity of the algorithm to compute the topology of the space curve is $\tilde{\mathcal{O}}_B(d^{37}\tau)$. □

We need to mention that in (Daouda et. al. (2012)), the authors gave a complexity bound $\tilde{\mathcal{O}}_B(d^{21}\tau)$ under the assumption that the algebraic space curve is in a generic position. For our method, when computing an s with a better bitsize, we can get a better complexity for the algorithm. We will improve the complexity of the algorithm in our future work.

3.4. Error control of the approximation space curves

We will show how to control the error of the approximation space curve in this subsection.

Theorem 9. Use the notations as before. If we approximate the plane curves $h = 0$ and $\bar{h} = 0$ with errors ϵ_1, ϵ_2, respectively, the error of each coordinate of the approximating curve of the algebraic space curve $f \wedge g$ is bounded by $\max(\epsilon_1, \frac{\epsilon_1+\epsilon_2}{s})$, and the Housdorff distance error of the approximating curve is bounded by $\frac{\sqrt{s^2\epsilon_1^2+(\epsilon_1+\epsilon_2)^2}}{s}$.

Proof. Let $C : (x, \tilde{y}(x))$ $(C_i : (x, \tilde{y_i}(x)))$ be the regular curve segment of $h = 0$ $(\bar{h} = 0)$ and $\bar{C} : (x, w(x))$ $(\bar{C_i} : (x, w_i(x)))$ its approximation curve, $x \in [x_0, x_1]$. C_i corresponds to C. Let $S_i : (x, \tilde{y}(x), \tilde{z}(x))$ (exact representation) and $\bar{S_i} : (x, w(x), \frac{w_i(x)-w(x)}{s}), x \in [x_0, x_1]$ be a space regular curve segment and its approximation. From the condition, we have $e(C, \bar{C}) < \epsilon_1, e(C_i, \bar{C_i}) < \epsilon_2$. The error here is defined by (8). Let us consider the three coordinates of one part of the approximation curve $S : (x, w(x), \frac{w_i(x)-w(x)}{s}), x \in [x_0, x_1]$.

The errors of the first and second coordinates are 0 and ϵ_1, respectively. For the third coordinate, we have $\tilde{z}(x) = \frac{\tilde{y_i}(x)-\tilde{y}(x)}{s}$. Thus

$$|\tilde{z}(x) - \frac{w_i(x) - w(x)}{s}| = |\frac{\tilde{y_i}(x) - \tilde{y}(x)}{s} - \frac{w_i(x) - w(x)}{s}| \leq \frac{|\tilde{y_i}(x) - w_i(x)| + |\tilde{y}(x) - w(x)|}{s} \leq \frac{\epsilon_1 + \epsilon_2}{s}.$$

So the third coordinate is bounded by $\frac{\epsilon_1+\epsilon_2}{s}$ from (8). From the definition of Hausdorff distance (6), we have the Hausdorff distance of S_i and $\tilde{S_i}$:

$$e(S_i, \tilde{S_i}) = \max_{P\in S_i} \min_{P'\in \tilde{S_i}} d(P, P') \leq \max_{P\in S_i, P'\in \tilde{S_i}, P'_x = P_x} d(P, P') < \sqrt{\epsilon_1^2 + ((\epsilon_1 + \epsilon_2)/s)^2} = \frac{\sqrt{s^2\epsilon_1^2 + (\epsilon_1 + \epsilon_2)^2}}{s}.$$

This ends the proof. □

If the required precision for the approximation curve is ϵ, we can approximate the plane algebraic curves $h=0$ and $\bar{h}=0$ with precision $\frac{s}{\sqrt{s^2+4}}\epsilon$ from the theorem.

3.5. G^1-continuous rational approximation space curve

We will derive approximation space curve from plane approximation curve. And we will re-parameterize the non-rational parametric curve into rational ones. Thus the obtained approximation space parametric curves are G^1-continuous and rational.

LEMMA 10. Use the notations as before. If we approximate the plane curves $h=0$ and $\bar{h}=0$ with C^1-continuous parametric curve, the approximation curve of the algebraic space curve $f \wedge g$ is C^1-continuous.

Proof. Let $(x, w_1(x)), (x, w_2(x)), x \in [x_0, x_1]$ be two corresponding approximation curves of the regular curve segments of $h=0$ and $\bar{h}=0$ and $w_1(x), w_2(x)$ are C^1-continuous in $[x_0, x_1]$. We can obtain the approximating curve of the space regular curve segment: $S: (x, w_1(x), \frac{w_2(x)-w_1(x)}{s}), x \in [x_0, x_1]$. The tangent direction of S at any x is $(1, \frac{\partial w_1}{\partial x}, (\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial x})/s)$. Thus S is C^1-continuous. □

When re-parameterizing the approximation space regular curve segments into rational ones, we need to know the tangent directions of the endpoints of space regular curve segments. For the endpoints corresponding to non-VT points, we can directly get it from the tangent directions of the plane curves. For the endpoints corresponding to VT points, we can get the tangent directions as follows. At first, we assume that $(x, w_1(x)), (x, w_2(x)), x \in [x_0, x_1]$ are parametric plane regular curve segments of exact algebraic regular curve segments $(x, \tilde{y}_1(x)), (x, \tilde{y}_2(x)), x \in [x_0, x_1]$ and x_0 corresponds to a VT point. The exact tangent direction of the algebraic space regular curve segment at x_0 is $(1, \tilde{y}_1'(x_0), (\tilde{y}_2'(x_0)-\tilde{y}_1'(x_0))/s)$ from the parametric representation. Note that $(1, \infty)$ corresponds to $(0, 1)$ for plane regular curve segments. So for the approximation tangent direction at x_0: $(1, \frac{\partial w_1(x_0)}{\partial x}, (\frac{\partial w_2(x_0)}{\partial x} - \frac{\partial w_1(x_0)}{\partial x})/s)$, if $\frac{\partial w_1(x_0)}{\partial x}$ is larger than (or less than) some given value, for example, 100 (or -100), we can reset the tangent direction as $(0, 1, (\frac{\partial w_2(x_0)}{\partial x} - \frac{\partial w_1(x_0)}{\partial x})/(s\, \frac{\partial w_1(x_0)}{\partial x}))$. Moreover, if $(\frac{\partial w_2(x_0)}{\partial x} - \frac{\partial w_1(x_0)}{\partial x})/(s\, \frac{\partial w_1(x_0)}{\partial x})$ is larger than (or less than) some given value, we can set the tangent direction as $(0, 0, \pm 1)$. So the tangent directions at x_0 is as $(0, 1, k), k \neq 0$ or $(0, 0, \pm 1)$.

Reparametrization of space curve. If the tangent direction at x_0 is $(0, 1, k)$, we can re-parameterize the space curve segment with the form

$$\mathbf{P}(t) = (\frac{a_1\, t^2 + b_1\, t + c_1}{d_1\, t + 1}, t, \frac{a_2\, t^2 + b_2\, t + c_2}{d_2\, t + 1} + \frac{c_3}{d_3\, t + 1}), \; t \in [0, 1], \tag{12}$$

such that it is G^1-continuous with other regular curve segments at the endpoints. Assume that the two endpoints are $(x_i, y_i, z_i), i = 0, 1$ and the given tangent directions at two endpoints are $(x_i', y_i', z_i'), i = 0, 1$. Thus $x_0' = 0$. Here for simplicity, we assume that $y_0 = 0, y_1 = 1$ since we can set $t = \frac{y - y_0}{y_1 - y_0}$. Bisecting the regular curve segment ensures that $y_1' \neq 0$ since the regular curve segment is monotonous.

We require that the parametric space curve satisfying G^0 and G^1 conditions at the two endpoints. So we have eight valid equations from the following equations.

$$\mathbf{P}(t)|_{t=0} = (x_0, y_0, z_0), \mathbf{P}(t)|_{t=1} = (x_1, y_1, z_1), \frac{\partial \mathbf{P}(t)}{\partial t}|_{t=0} = (0, 1, p), \frac{\partial \mathbf{P}(t)}{\partial t}|_{t=1} = \frac{1}{y_i'}(x_1', y_1', z_1').$$

Solving them, we have one solution as below.

$$\begin{aligned}
a_1 &= \frac{{x_0}^2 - 2\,x_0x_1 + {x_1}^2}{-x_1 + x_1' + x_0},\\
a_2 &= \frac{1}{d_3}(z_1'd_3 + z_1'd_2d_3 - z_1d_3 + d_3z_0 + z_0' + z_1'd_2 - z_1d_2 + d_2z_0 + z_1' - 2\,z_1 + 2\,z_0),\\
b_1 &= -\frac{x_0\,(2\,x_0 - 2\,x_1 + x_1')}{-x_1 + x_1' + x_0},\\
b_2 &= -\frac{1}{d_3^2}(-z_1d_2 + d_2z_0 + z_1'd_2 - 2\,z_1d_2d_3 + 2\,d_2z_0d_3 + z_1'd_2{d_3}^2\\
&\quad +2\,z_1'd_2d_3 - z_1d_2{d_3}^2 + 4\,d_3z_0 + 2\,z_0 - 2\,z_1 + z_0' + z_1'{d_3}^2 + 2\,z_1'd_3\\
&\quad -4\,z_1d_3 + 2\,d_3z_0' + z_1' - 2\,z_1{d_3}^2 + 2\,{d_3}^2z_0),\\
c_1 &= x_0,\\
c_2 &= -\frac{1}{{d_3}^2(-d_3 + d_2)}(-2\,z_1d_2d_3 + z_1'd_2{d_3}^2 + 2\,z_1'd_2d_3 + z_1' + z_0' - 2\,z_1\\
&\quad +2\,z_0 - z_1d_2 - 4\,z_1d_3 + z_1'd_2 + z_1'{d_3}^2 + 2\,z_1'd_3 - z_1d_2{d_3}^2 + 2\,d_2z_0d_3\\
&\quad +d_2z_0 + 4\,d_3z_0 + 2\,{d_3}^2z_0 + 2\,d_3z_0' + {d_3}^2z_0' - 2\,z_1{d_3}^2 + {d_3}^3z_0),\\
c_3 &= \frac{1}{{d_3}^2(-d_3 + d_2)}(-2\,z_1d_2d_3 + z_1'd_2{d_3}^2 + 2\,z_1'd_2d_3 + z_1' + z_0' - 2\,z_1\\
&\quad +2\,z_0 - z_1d_2 - 4\,z_1d_3 + z_1'd_2 + z_1'{d_3}^2 + 2\,z_1'd_3 - z_1d_2{d_3}^2 + d_2{d_3}^2z_0\\
&\quad +2\,d_2z_0d_3 + d_2z_0 + 4\,d_3z_0 + 2\,{d_3}^2z_0 + 2\,d_3z_0' + {d_3}^2z_0' - 2\,z_1{d_3}^2),\\
d_1 &= -\frac{2\,x_0 - 2\,x_1 + x_1'}{-x_1 + x_1' + x_0},
\end{aligned}$$

where d_2, d_3 are free variables. At first, we require that $x_1' - x_1 + x_0 \neq 0$, $d_3(d_2 - d_3) \neq 0$ since they are denominators. Second, we require that $d_i\,t + 1 = 0 (i = 1, 2, 3)$ in t has no root in $[0, 1]$, that is, $d_i > -1$. For $i = 1$, we have equal conditions:

$$(x_0 - x_1)(x_0 - x_1 + x_1') < 0. \tag{13}$$

Since the given planar regular curve segment is monotonous (w.r.t. both x and y), the first condition and (13) hold directly. We can choose proper d_2, d_3 such that conditions hold.

When the tangent direction is $(0, 0, 1)$, we set the parametric regular curve segment as

$$\mathbf{P}(t) = (\frac{a_1\,t^2 + b_1\,t + c_1}{d_1\,t + 1}, \frac{a_2\,t^2 + b_2\,t + c_2}{d_2\,t + 1} + \frac{c_3}{d_3\,t + 1}, t),\ t \in [0, 1],$$

and solve a similar equation system to get the parametric regular curve segments.

The left problem is to control the precision. Let ϵ be the required precision for the whole approximation parametric curve. If the non-rational parametric curve $\mathcal{S}_1 : (x, p(x), q(x)), x \in [x_0, x_1]$ approximates the regular curve segment of algebraic space curve S with precision $\epsilon/2$, and the new rational parametric curve $\mathcal{S}_2 : (x(t), y(t), z(t))$ approximate $\mathcal{S}_1 : (x, p(x), q(x))$ with precision $\epsilon/2$, then $\mathcal{S}_2$ approximate S with precision ϵ.

We need to control the approximation precision of $\mathcal{S}_2$ to $\mathcal{S}_1$. In (Shen et. al. (2011)), the authors consider the approximation of 3-D parametric curve with rational Bézier curves with correct topology. For our problem, we need rational curve. But the technique in the paper to ensure the correct topology can be used for our problem. For any fixed $x^0 \in [x_0, x_1]$, we can derive a univariate polynomial equation in t of degree 2 by $p(x^0) = y(t)$. Solving it, we have two real solutions (the solutions do exist). Choose the one such that $x(t)$ close to x^0, say t^0. Denote the distance between $(x^0, p(x^0), q(x^0))$ and $(x(t^0), y(t^0), z(t^0))$ as $D(x_0)$. From the definition (6), we can find that $\max_{x^0\in[x_0,x_1]} D(x^0) \geq e(\mathcal{S}_1, \mathcal{S}_2)$ is an upper bound of the Hausdorff distance of $\mathcal{S}_1$ and $\mathcal{S}_2$. We can choose some sample points to estimate the error between $\mathcal{S}_1$ and $\mathcal{S}_2$.

Thus, in the end, we get a G^1-continuous piecewise rational approximation space curve under a given precision.

When we approximate a regular curve segment containing a VT point in practice, we usually select a short distance for it since the error control is much easier.

4. Algorithm and examples

In this section, we will give the main algorithm to approximate algebraic space curves and use some non-trivial examples to illustrate the effectivity of our algorithm.

Algorithm 2. The inputs are $f, g \in \mathbb{Q}[x, y, z]$ such that $\gcd(f, g) = 1$ and satisfying the two assumptions, a bounding box $\mathbf{B} = [X_1, X_2] \times [Y_1, Y_2] \times [Z_1, Z_2]$ and an error bound $\epsilon > 0$. The outputs are piecewise rational parametric regular curve segments $\mathbb{C}_i := \{(x, y_i(x), z_i(x))(\text{ or } (x_i(y), y, z_i(y)), a_i \leq x(\text{ or } y) \leq b_i, (i = 1, ..., N)\}$, which give a G^1-continuous approximation to $f \wedge g$ in $\mathbf{B}$ with precision ϵ.

(1) Topology determination and regular curve segmentation of the plane curve defined by $\mathcal{C}_1 : \pi_z(f \wedge g)$.
(2) Compute a rational number s as mentioned in Section 3.1.
(3) Let $\varphi_s : (x, y, z) \to (x, y + s\, z, z)$. Topology determination and regular curve segmentation of the plane curve defined by $\mathcal{C}_2 : \pi_z(\varphi_s(f) \wedge \varphi_s(g))$.
(4) Find out the correspondence between the regular curve segments of $\mathcal{C}_1$ and $\mathcal{C}_2$.
(5) Approximate the regular curve segments without VT point of $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\epsilon_0 < \frac{s}{\sqrt{s^2+4}}\epsilon$ and the ones with VT point with precision $\epsilon_0 < \frac{s}{2\sqrt{s^2+4}}\epsilon$.
(6) Recover the space approximation regular curve segments of $f \wedge g$ with formula (11).
(7) Re-parameterize the non-rational approximation curves to rational approximation curves under the error control if there exist.
(8) Output the piecewise approximation regular curve segments.

We will show several examples to illustrate our algorithm.

Example 1. Consider the algebraic space curve defined by the system $\{f, g\} = \{x^2 + y^2 + z^2 - 4, (z - 1)\,(x^2 + y^2 - 3\,z^2)\}$. In fact, they are two plane circles with $z = \pm 1$ as shown in Fig. 8 (green ones). The space curve is not irreducible, not regular, and not in a generic position. We will approximate it with rational curves under precision 10^{-2}. In this example, we use floating numbers to replace rational numbers in the computation.

Following the algorithm above, we have

(1) Compute the resultant of f, g w.r.t. z, we have $\mathbb{C}_1 : h = x^2+y^2-3 = 0$, as the red circle $PWQTP$ in Fig. 8. We split $\mathbb{C}_1$ into eight regular curve segments with x-coordinates $[-1.732, -1.0, 0, 1.0, 1.732]$. Note that $(\pm 1.732, 0)$ correspond to VT points.

(2) Since $\mathtt{Res}_y(h, h_y) = x^2 - 3$, we can obtain $\alpha_0 = -1.732, \alpha_1 = 1.732$. And we can get $r = 3.464$. We can get $R = 1.0$ with g. We set $s = 1 < \frac{r}{2R} = 1.732$.

(3) Compute the resultant of $\varphi(f) = f(x, y+z, z), \varphi(g) = g(x, y+z, z)$ w.r.t. z, we have $\mathbb{C}_2 : \bar{h} = (x^2 + y^2 - 2 + 2\,y)(-2 + x^2 - 2\,y + y^2) = 0$, as two blue circles $P_1W_1Q_1T_1P_1, P_2W_2Q_2T_2P_2$ in Fig. 8. Since $\pi_y(\bar{h}) = (x^2-3)(x^2-2) = 0$, we split $\mathbb{C}_2$ into 16 regular curve segments at $x = \{-1.732, -1.414, 0, 1.414, 1.732\}$.

(4) As shown in Fig. 8, the critical points of $\mathbb{C}_1$ are P, Q. Choose a vertical line which intersect $\mathbb{C}_1$ at W, T. $K(K = P, Q, W, T)$ are points on $\mathbb{C}_1$ and K_1, K_2 are corresponding points of K on $\mathbb{C}_2$. Consider $W(0, 1.732), W_1(0, 2.732), W_2(0, 0.732)$ for example. We can find that W_1, W_2 are on the line $x = 0$ in a neighborhood with radius 1.732 centered at W. So we can conclude that W_1, W_2 correspond to W with local generic position method. The correspondence of other points are similar.

(5) Approximate $\mathbb{C}_1, \mathbb{C}_2$ respectively. In order to derive the required precision 10^{-2}, we we use precision $\epsilon_1 = 0.0044 < \frac{1}{\sqrt{1^2+4}}10^{-2}$ for the regular curve segments of $\mathbb{C}_1, \mathbb{C}_2$ without VT point(s), and we use precision $\epsilon_2 = 0.0022 < \frac{1}{2\sqrt{1^2+4}}10^{-2}$ for the regular curve segments with VT point. Consider a regular curve segments on $\mathbb{C}_1$, $(-1.732, 0)$, $(-1.60, 0.663)$ are the endpoints for the one, denoted as C_1. And it has a VT point. $(-1.60, 0.663), (-1.40, 1.012)$ are endpoints for the other, denoted as C_2. And it has no VT point. The approximation of C_1 is

$$(x, 1.0\,\sqrt{-x^2 + 0.000000464\,x + 3.000}), x \in [-1.732, -1.600]$$

and the error is very small. The approximation for C_2 is

$$(x, 0.611\,x + 2.311 - 0.127/(0.507\,x + 1.0)), x \in [-1.60, -1.40]$$

and the error is $0.0004 < \epsilon_1$. For the regular curve segments on $\mathbb{C}_2$ with endpoints: $[(-1.732, 1.0), (-1.60, 1.663)]$, denoted as C_3 and it has a VT point. $(-1.60,\ 1.663)$, $(-1.414, 2.0)$ are endpoints for the other, denoted as C_4, without VT point. Similarly as C_1, the approximation for C_3 is

$$(x, 1.0 + 1.000\,\sqrt{-x^2 + 0.000000466\,x + 3.000}, x \in [-1.732, -1.60].$$

The approximation for C_4 is

$$(x, 0.630\,x + 3.324 - 0.127/(0.509\,x + 1.0)), x \in [-1.60, -1.414]$$

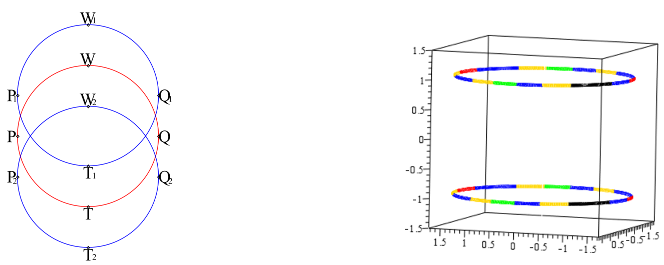

Fig. 8. Projection curves and approximation curve in Example 1.

and the error is $0.0002 < \epsilon_1$. We can find that parts of C_1, C_2 and C_3, C_4 are correspondent.

(6) Recover the approximation space curves of $f \wedge g$ by the formula $z = \frac{y_2(x)-y_1(x)}{s}$. The space regular curve segment corresponding to C_1 and C_3, we have its approximation parametric space regular curve segment for $x \in [-1.732050808, -1.60]$:

$$(x, 1.000\sqrt{-x^2 + 0.000000464\,x + 3.000},$$
$$1.0 + 1.000\sqrt{-x^2 + 0.000000466\,x + 3.000} - 1.000\sqrt{-x^2 + 0.000000464\,x + 3.000}).$$

The approximation space curve is not rational, denoted as S_1. The approximation corresponding to C_2, C_4 for $x \in [-1.60, -1.414]$, denoted as S_2, is $(x, 0.611\,x + 2.311 - \frac{0.127}{0.507\,x+1.0}, 0.0195\,x + 1.013 - \frac{0.127}{0.509\,x+1.0} + \frac{0.127}{0.507\,x+1.0})$.

(7) We will re-parameterize S_1 into rational one. At first, we can find that the y coordinate of S_1 changes from 0 to 0.663. Its two endpoints are $P_0(-1.732, 0, 1.0)$, $P_1(-1.60, 0.663, 1.0)$. The tangent direction of S_2 at P_2 is $(1, 2.412, 0.0)$. By approximating the tangent direction of S_1 at P_1, we have $(1, 283.078, 0.0)$. And there is another regular curve segment which shares a same tangent direction with S_1 at P_1. Taking their average value, we can set the tangent direction of S_1 at P_1 as $(0, 1, 0)$. Using Formula (12), we can easily obtain the rational approximation regular curve segment for S_1 is $(-2.412\,y - 22.719 + \frac{20.987}{-0.115\,y+1}, y, 1.0)$. The error in x-direction is $0.002 < \epsilon_2$ (We take 19 sample points besides endpoints to compute the error). So the approximation rational curve satisfies the error requirement.

(8) Output the piecewise approximation curves.

Example 2. Approximate the algebraic space curve defined by $f = g = 0$, where $f = x^2 + y^2 + z^2 - 4, g = \left(x^2 + y^2 + 2\,y - z^2\right)(z - x - 4\,y)$. It is a space curve with singular point. The approximation space curve is as the left part of Fig. 9 and the error is 0.013. The color differs the different approximating space regular curve segments.

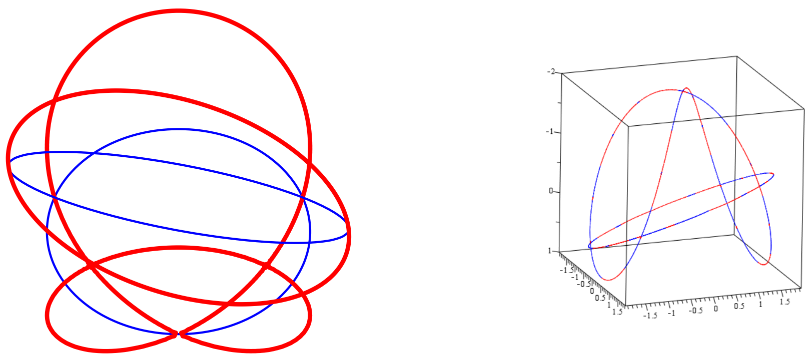

Fig. 9. The projection curves and the approximation curve in Example 2.

Example 3. In this example, we will approximate the algebraic space curve defined by $f = g = 0$ inside $\mathbf{B} = [-2, 2] \times [-2, 2] \times [-2, 2]$ with error $\epsilon = 0.014$, where $f = 95 - 94\,x^3 + 64\,x^2y + 28\,x^2z - 61\,x^2 + 69\,xy^2 - 53\,xyz - 59\,xy + 28\,xz^2 - 15\,xz - 83\,x - 3\,y^3 + 59\,y^2z + 49\,y^2 +$

$4\,yz^2+11\,yz+5\,y-81\,z^3-8\,z^2-9\,z, g = 49+7\,x^3-46\,x^2y+87\,x^2z+94\,x^2+73\,xy^2+93\,xyz-3\,xy-27\,xz^2+56\,xz+70\,x+72\,y^3-37\,y^2z-20\,y^2+79\,yz^2-78\,yz-3\,y+94\,z^3+30\,z^2+47\,z$.

In this example, we focus only on the space curve segments of the whole algebraic space curve inside a given cube. So we need to compute the intersection of the space curve with the six faces of the cube **B**. When we compute s, we can use only the α_i's inside the cube. In order to illustrate the example in an easy way, we use floating number to represent the points. But in the implementation, we use an isolating interval to represent a point.

At first, we compute the square free part of $\texttt{Res}_z(f, g)$, denoted as h. Then we compute the intersection points of $\{f(x, y, 2), g(x, y, 2)\}$ and $\{f(x, y, -2), g(x, y, -2)\}$ inside $[-2, 2] \times [-2, 2]$. We get one point $(0.224, -1.818)$, as the box point shown on the projection curve in Fig 10. We compute the intersection points of $h = 0$ and $y = \pm 2$, $\frac{\partial h}{\partial y} = 0$. We get the x-coordinates of all these points as $0.224, 0.427, 0.784, 1.234$, where 0.784 corresponds to an x-critical point. The reason why we compute all these points is that we need to decide which curve segment of $h = 0$ corresponding to the space curve segments inside **B**.

We also need to compute the points of the algebraic space curve on the faces $x = \pm 2, y = \pm 2$. Denote the set of these points as $\mathcal{P}$.

Using $x = 0.784$ to compute r, we can choose $r = 2$. We choose upper bound of the absolute value of the real roots of $\gcd(f(\alpha_i, \beta_{i,j}, z), g(\alpha_i, \beta_{i,j}, z)) = 0$ as R. We can get it by the interval polynomials of f, g on the isolating intervals of the plane points. We can get $R > 1$. Thus we can set $s = 1$ by Equation (10).

Then we compute $\bar{h} = \texttt{Res}_z(\varphi(f), \varphi(g))$ and the topology of $\bar{h} = 0$. We need to find the corresponding points of $\mathcal{P}$ on $\bar{h} = 0$, as the diamond points shown in the projection curve in Fig. 10. We also need to select the curve segments of $\bar{h} = 0$ which correspond to the space curve segments in **B**.

The left thing is to find the correspondence of $h = 0$ and $\bar{h} = 0$ for the curve segments we are interested, and the approximation of the space curve segments. We ignore the detail for this part. It is similar to Example 1.

The projection curves and the approximation space curve are shown in Fig. 10.

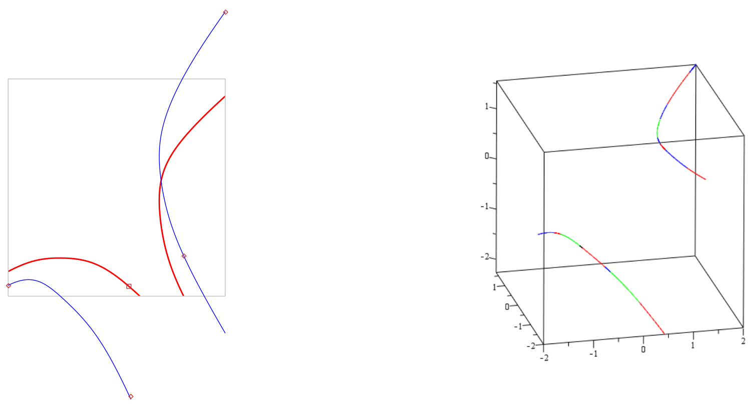

Fig. 10. The projection curves and the approximation curve in Example 3.

5. Conclusion

We introduce a local generic position method to compute the topology of an algebraic space curve. The bit complexity of computing the topology of an algebraic space curve is $\tilde{\mathcal{O}}(d^{37}\tau)$. Based on the topology of a space curve, we present an algorithm to approximate algebraic space curve by piecewise rational curves with correct topology and under any given precision. The method is effective.

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