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Large time behavior for the nonstationary Navier–Stokes flows in the half-space

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Asymptotic behavior of higher-order spatial derivatives in L^r is established for incompressible Navier–Stokes flows in the half-space, which is a long-time unsolved problem. The main tools employed for solving this problem are the nonstationary Stokes solution formula, and the a priori estimates of the steady Stokes system in the half-space. Another main result is devoted to studying time L^1-decay, and a partial answer to this open problem is given by means of a crucial and new estimate for the Stokes flow. Two Theorems 1.1, 1.2 in this article can be regarded as a great improvement and powerful extension of the work in [6,38].

1. Introduction and main results

This paper studies the behavior as $t \longrightarrow \infty$ of the nonstationary incompressible Navier–Stokes flows in the half-space $\mathbb{R}^n_+$, $n \geq 2$:

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$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \nabla\cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is the upper half-space of $\mathbb{R}^n$; the functions $u = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$ and $p = p(x,t)$ denote, respectively, unknown velocity and pressure; while the given initial velocity $u_0(x)$ is assumed to satisfy a *compatibility condition*: $\nabla\cdot u_0 = 0$ in $\mathbb{R}^n_+$ and the normal component of u_0 equals zero on $\partial\mathbb{R}^n_+$; and

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla = (\partial_1, \partial_2, \cdots, \partial_n), \quad \partial_j = \frac{\partial}{\partial x_j}\ (j = 1, 2, \cdots, n),$$

$$\Delta u = \sum_{j=1}^n \partial_j\partial_j u, \quad (u\cdot\nabla)u = \sum_{j=1}^n u_j\partial_j u, \quad \nabla\cdot u = \sum_{j=1}^n \partial_j u_j.$$

Throughout this paper, $C_0^\infty(\mathbb{R}^n_+)$ denotes the set of all C^∞ real functions with compact support in $\mathbb{R}^n_+$, and

$$C^\infty_{0,\sigma}(\mathbb{R}^n_+) = \{\phi = (\phi_1, \cdots, \phi_n) \in C_0^\infty(\mathbb{R}^n_+)\ : \nabla\cdot\phi = 0 \ \text{ in } \ \mathbb{R}^n_+\};$$

$L^q_\sigma(\mathbb{R}^n_+)$ $(1 < q < \infty)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to $\|\cdot\|_{L^q(\mathbb{R}^n_+)}$, where $L^q(\mathbb{R}^n_+)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^q(\mathbb{R}^n_+)$ is denoted by $\|u\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |u(x)|^q dx)^{\frac{1}{q}}$ if $1 \le q < \infty$; and $\|u\|_{L^\infty(\mathbb{R}^n_+)} = ess \sup_{x\in\mathbb{R}^n_+} |u(x)|$; $\|x_n^\alpha u(t)\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |x_n^\alpha u(x,t)|^q dx)^{\frac{1}{q}}$ if $1 \le q < \infty$; $\|x_n^\alpha u(t)\|_{L^\infty(\mathbb{R}^n_+)} = ess \sup_{x\in\mathbb{R}^n_+} |x_n^\alpha u(x,t)|$, $\alpha \ge 0$. By symbol C, it means a generic positive constant which may vary from line to line.

$u \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^n_+)) \bigcap L^2_{loc}([0,\infty); W_0^{1,2}(\mathbb{R}^n_+))$ is called a weak solution of (1.1) with the initial data $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ if u satisfies problem (1.1) in the distribution sense. Moreover, the energy inequality holds for almost all $t \in [0,\infty)$ including $t = 0$: $\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} ds \le \|u_0\|^2_{L^2(\mathbb{R}^n_+)}$. Further, we call u is a strong solution of (1.1) if u satisfies the regularity criteria: $u \in L^q(0,\infty; L^r(\mathbb{R}^n_+))$ with $\frac{2}{q} + \frac{n}{r} \le 1$, $2 \le q < \infty$, $n < r \le \infty$.

There is abundant literature in studying properties of solutions to the Navier–Stokes equations in the whole space. The existence of global weak solutions in the energy space goes back to Leray [30], and the uniqueness of these solutions is only known in space dimension two. Meanwhile it is well known that smooth solutions are global in dimension two and for higher dimensions when the data are small in some critical spaces. By making use of the (L^p, L^q)-estimates for the solutions of the heat equations, Kato [29] proved some results on the existence and decay properties of local and global solutions of the Cauchy problem for the Navier–Stokes equations. Caffarelli, Kohn and Nirenberg

[14] proved that the one-dimensional Hausdorff measure is zero for the set of the possible space–time singular points, for the suitable weak solutions to the 3D Navier–Stokes equations; by means of different methods, Lin [31] verified the same conclusion. Their results bring about a deeper understanding on the regularity theory and narrowed the gap between what we can get from the global existence theory and what we need for the uniqueness and regularity of global weak solutions to the incompressible Navier–Stokes equations. For further results, see [4,17,18] and the references therein. Wu [43–45] considered some other important mathematical models related closely to problem (1.1), and obtained many nontrivial results.

Consider the Helmholtz decomposition [10]:

$$L^r(\mathbb{R}^n_+) = L^r_\sigma(\mathbb{R}^n_+) \oplus L^r_\pi(\mathbb{R}^n_+), \quad 1 < r < \infty,$$

with

$$L^r_\sigma(\mathbb{R}^n_+) = \{u = (u_1, u_2, \cdots, u_n) \in L^r(\mathbb{R}^n_+);\ \ \nabla \cdot u = 0,\ u_n|_{\partial \mathbb{R}^n_+} = 0\},$$
$$L^r_\pi(\mathbb{R}^n_+) = \{\nabla p \in L^r(\mathbb{R}^n_+);\ \ p \in L^r_{loc}(\overline{\mathbb{R}^n_+})\}.$$

Let $A = -P\Delta$ denote the Stokes operator in $\mathbb{R}^n_+$, where $P = P_r$ is the associated bounded projection: $L^r(\mathbb{R}^n_+) \longrightarrow L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$. Then (see [10]) the operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ in $L^r_\sigma(\mathbb{R}^n_+)$. So for each $u_0 \in L^r_\sigma(\mathbb{R}^n_+)$, the function $e^{-tA}u_0$ gives a unique solution of Stokes system in $L^r_\sigma(\mathbb{R}^n_+)$, i.e., problem (1.1) with $u \cdot \nabla u$ deleted, with the initial value u_0.

The decay properties for the Navier–Stokes flows have been considered by many mathematicians in recent years, and lots of interesting results are obtained. The problem of L^2-decay for weak solutions of the Navier–Stokes equations was first raised by Leray [30] in the case of the Cauchy problem in $\mathbb{R}^3$. Schonbek [33,36] attacked this problem and succeeded for the first time in showing the existence of weak solutions with explicit decay rate. In addition, Bae and Jin [6] showed the weighted energy inequality for weak solutions of the Cauchy problem under some conditions on the initial data. For further results, please refer to [11–13,34,35] and the references therein. Here it should be mentioned that Schonbek and Wiegner [38] studied the decay of higher-order norms of the Navier–Stokes flows in $\mathbb{R}^n$ in terms of its energy decay, where the proof of the results relies upon some energy estimates for higher-order spatial derivatives, the known L^∞-decay estimate and the Fourier splitting method due to Schonbek [37]. Constantin and Wu [15] focused on the decay properties for a class of quasi-geostrophic equations.

However, all of the above-mentioned works in the whole space rely on the theory of Fourier transform, in treating the nonlinear term $u \cdot \nabla u$, and therefore the arguments developed there cannot be directly applied to the case of the half-space $\mathbb{R}^n_+$ with noncompact boundary $\partial\mathbb{R}^n_+$. The difficulty comes from the lack of the weighted estimates with respect to pressure because of the appearance of the boundary. As will be shown below, the essential feature is different, these mentioned arguments on the decay results in the whole space $\mathbb{R}^n$ cannot be applied because the boundary is non-compact, and $\partial_n P \neq P\partial_n$ on the half-space $\mathbb{R}^n_+$, which produces many incredible difficulties in dealing with the asymptotic behavior of the higher-order derivatives for the strong solution of (1.1). Using the fractional powers of the Stokes operator in general L^r spaces, Borchers and Miyakawa [10] established the L^2-decay for weak solutions of (1.1). Bae and Choe [3] studied the asymptotic behavior for solutions of (1.1) in $L^q(\mathbb{R}^n_+)$ with $1 < q < \infty$. Under some constraint conditions on the initial data, Han [23,24] showed the various weighted decays of $\|\nabla^m u(t)\|_{L^r(\mathbb{R}^n_+)}$ for the strong solution u of (1.1), where $m = 0, 1, 2$, $1 \leq r \leq \infty$, and $r \neq 1$ if $m = 0$. For the related topics in the case of exterior domains, please refer to the literature [5,7–9,26–28].

To proceed, it is necessary to state a small-data global existence of classical solution in the half-space, which is refereed to Theorem 3.2 in the explicit reference [19].

Theorem 1.0. *(See [19].) Let $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$. Then there exists a number $\eta_0 > 0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)} \leq \eta_0$ (smallness condition is unnecessary if $n = 2$), problem (1.1) admits a unique strong solution (u, p).*

Our aim in this article is to establish the asymptotic properties of higher-order norms of the Navier–Stokes flows in Theorem 1.0 under some specific conditions on the initial velocities, which extends the results of [6,38] to the case of flows in the half-space. The following result (i.e. Theorem 1.1) shows that the decay rate is optimal for $n \geq 3$ in the sense that it coincides with that of the fundamental solution to the heat equation, which is motivated by the work in Schonbek and Wiegner [38], where they considered the simpler case of the whole space, and their method, however, depends on the Fourier transform and on the commutativity between projection and differential operators, neither of which seems to be valid for the problem in the half-space $\mathbb{R}^n_+$ with boundary $\partial\mathbb{R}^n_+$.

Theorem 1.1. *Suppose $u_0 \in L^1(\mathbb{R}^n_+) \bigcap L^q_\sigma(\mathbb{R}^n_+)$ for all $1 < q < \infty$, $n \geq 2$, then for every integer $k \geq 1$, there exists $t_k > 0$ such that for $1 < r \leq \infty$, $\epsilon \in (0, \frac{1}{2})$ and $t \geq t_k$, if u_0 is small in $L^n_\sigma(\mathbb{R}^n_+)$, the strong solution (u, p) given in Theorem 1.0 satisfies*

$$\|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^{1+k} p(t)\|_{L^r(\mathbb{R}^n_+)} \leq \begin{cases} Ct^{-\frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})} & \text{if } n \geq 3, \\ C_\epsilon t^{\epsilon - \frac{2+k}{2} - (1-\frac{1}{r})} & \text{if } n = 2. \end{cases} \tag{1.2}$$

Further, if u_0 satisfies $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, then for $n \geq 2$ and $t \geq t_k$

$$\|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^{1+k} p(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2} - \frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})}, \quad 1 < r \leq \infty. \tag{1.3}$$

In (1.2), (1.3), the constants $C = C(k, r, n, u_0)$ and $C_\epsilon = C(\epsilon, k, r, n, u_0)$.

Remark. To our knowledge, it is the first time to give the decay results for the higher-order spatial derivatives of the strong solution of (1.1), which is a challenging unsolved question for a long time in the case of the half-space. The main difficulty lies that the strong singularity appears in estimating $\|\nabla^\ell u(t)\|_{L^r(\mathbb{R}^n_+)}$ with $\ell \geq 2$ by using Solonnikov's solution formula or the integral equality of the strong solution u of (1.1). By a clever use of the Ukai's solution formula, the strong singularity appearing from $\|\nabla^\ell u(t)\|_{L^r(\mathbb{R}^n_+)}$ with $\ell = 2$ can be avoided, see [22,23]. However, this method employed in [22,23] does not work for $\ell \geq 3$ because the singularity is really too strong. As will be seen in Section 2, such difficulty is overcome by applying regularity estimates of the steady Stokes system. In addition, it should be pointed out that the long-time decays of $\|\nabla^\ell u(t)\|_{L^r(\mathbb{R}^n_+)}$ ($\ell = 0, 1, 2$) have been established in [22] under the assumptions of Theorem 1.1, and that is why we require integer $k \geq 1$ in the estimates (1.2), (1.3).

The following weighted results are inspired by the work in Bae and Jin [6], where the weighted decay estimates of the Navier–Stokes flows in the whole space are established.

Theorem 1.2. *Assume $u_0 \in L^1(\mathbb{R}^n_+) \bigcap L^q_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$ for all $2 < q < \infty$ satisfies $x_n u_0$, $(1 + x_n)\nabla u_0 \in L^2(\mathbb{R}^n_+)$. Let $1 < r \leq \infty$, $0 < \beta < 1$. Then for every integer $k \geq 1$, there exists $\bar{t}_k > 0$ such that for $n \geq 3$ and $t \geq \bar{t}_k$, if u_0 is small in $L^n_\sigma(\mathbb{R}^n_+)$, the strong solution (u, p) given in Theorem 1.0 satisfies*

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq \begin{cases} \overline{C} t^{-\min\{\frac{n}{2}-\frac{1}{2}, \frac{3}{2}-\frac{\beta}{2}\} - \frac{n}{2}(1-\frac{1}{r})} & \text{if } k = 1, \\ \overline{C}_\epsilon t^{-\min\{\frac{n}{2}-\epsilon, \frac{2+k}{2}-\frac{\beta}{2}\} - \frac{n}{2}(1-\frac{1}{r})} & \text{if } k \geq 2. \end{cases} \tag{1.4}$$

Further, if u_0 satisfies $x_n u_0 \in L^1(\mathbb{R}^n_+)$ for $n \geq 2$, then for $n \geq 2$, $k \geq 1$ and $t \geq \bar{t}_k$

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq \overline{C}_\epsilon t^{-\min\{\frac{n}{2}-\epsilon, \frac{2+k}{2}-\frac{\beta}{2}\} - \frac{1}{2} - \frac{n}{2}(1-\frac{1}{r})}. \tag{1.5}$$

Here $\epsilon \in (0, \frac{1}{2})$ and $\overline{C} = \overline{C}(n, k, r, \beta, u_0)$, $\overline{C}_\epsilon = \overline{C}(n, k, r, \epsilon, \beta, u_0)$ in (1.4), (1.5).

Remarks. (1) In [23,24], the decays of $\|x_n^\beta \nabla^\ell u(t)\|_{L^r(\mathbb{R}^n_+)}$ have been established for large time $t > 0$, where $\ell = 0, 1, 2$, and $1 \leq r \leq \infty$ ($r \neq 1$ if $\ell = 0$). Whence Theorem 1.2 can be regarded as a great improvement of results in [23,24].

(2) Under the assumptions in Theorem 1.2, it is reasonable to ask whether the following decay estimates are valid for large time t

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq C t^{\frac{\beta}{2} - \frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})};$$

and if $x_n u_0 \in L^1(\mathbb{R}^n_+)$ for $n \geq 2$,

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{\frac{\beta}{2}-\frac{1}{2}-\frac{2+k}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{1.6}$$

Obviously, the estimate (1.6) is true for $n = 3$ and $k = 1$ by (1.5) with $0 < \epsilon < \frac{\beta}{2}$.

The main difficulty lies that we are nor sure whether the weighted estimate is true for the solution w of the stationary Stokes equations (see (2.20) below): let $1 < q < \infty$ and let $\ell \ge 0$ be an integer. Then

$$\begin{aligned} &\|x_n^\beta \nabla^{\ell+2} w\|_{L^q(\mathbb{R}^n_+)} + \|x_n^\beta \nabla^{\ell+1} \pi\|_{L^q(\mathbb{R}^n_+)} \\ &\le C(n, \ell, q, \beta)(\|x_n^\beta \nabla^\ell f\|_{L^q(\mathbb{R}^n_+)} + \|x_n^\beta \nabla^{\ell+1} g\|_{L^q(\mathbb{R}^n_+)}). \end{aligned} \tag{1.7}$$

The estimate (1.7) is valid for $\beta = 0$, see (2.21) below. Further, even if (1.7) is true, we still need to face and overcome another difficulty: how to handle the crucial estimate $\|x_n^\beta \partial_t^m u(t)\|_{L^q(\mathbb{R}^n_+)}$ $(m = 0, 1, \cdots)$ for the strong solution u of (1.1), which, at present, we have no ideas to deal with.

(3) The estimate of $\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)}$ in Theorem 1.2 is obtained by splitting the half-space into two parts: $\mathbb{R}^{n-1} \times (0, 1]$, $\mathbb{R}^{n-1} \times (1, \infty)$, and by making full use of the special structure in the Solonnikov's solution formula. However, checking the proof's process of Theorem 1.2, we readily find that the method employed in the proof of Theorem 1.2 does not work in the case of $\||x'|^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)}$, which up to now, is still an open question.

The following result focuses on the L^1-decay estimate of the Stokes flow and its spatial derivatives, and such decay estimate on the Stokes flow is also interesting by itself. Recently, people have paid much attention to such kind of topics. If $u_0 \in L^1_\sigma(\mathbb{R}^n_+)$ satisfies some additional conditions, Bae and Choe [3] showed that the decay rate of $\nabla e^{-tA} u_0$ in $L^q(\mathbb{R}^n_+)$ $(1 < q < \infty)$ could be controlled by t^{-1} times a constant. If the initial data u_0 lies in an appropriate weighted space, Bae [1,2] estimated the time decay rates of the gradient of Stokes solutions in $L^1(\mathbb{R}^n_+)$.

It should be pointed out that the investigation on the Stokes flow is essential, which plays a fundamental role in studying the time L^1-behavior for the Navier–Stokes equations. Actually, in order to establish L^1-decay for the Navier–Stokes solution u, people usually invoke the projection P onto the solenoidal vector fields to eliminate the pressure gradient ∇p in (1.1) and then transform problem (1.1) into the integral equation:

$$u(t) = e^{-tA} a - \int_0^t e^{-(t-s)A} P u(s) \cdot \nabla u(s) ds.$$

Note that for any $t \ge s$, $e^{-(t-s)A} P u(s) \cdot \nabla u(s)$ is also a Stokes flow with the initial value $P u(s) \cdot \nabla u(s)$.

Theorem 1.3. *Let $a = (a_1, a_2, \cdots, a_n) \in L^1(\mathbb{R}^n_+)$, $a_n|_{\partial \mathbb{R}^n_+} = 0$, $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \ge 2)$. Then for any $0 \le \alpha \le 1$ and $t > 0$*

$$\|e^{-tA}a\|_{L^1(\mathbb{R}^{n-1}\times(0,\sqrt{t}\,))} \leq Ct^{-\frac{\alpha}{2}} \int\limits_{\mathbb{R}^n_+} |x|^\alpha |a(x)|dx;$$

$$\|\nabla^k[e^{-tA}a]\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{k}{2}-\frac{\alpha}{2}} \int\limits_{\mathbb{R}^n_+} |y|^\alpha |a(y)|dy, \quad k=1,2,\cdots.$$

Remark. The decay of $\|e^{-tA}a\|_{L^1(\mathbb{R}^n_+)}$ has been established by Han [25] under the additional *tangential parity condition* on the initial data a. Such technical assumption in [25] is imposed to overcome the difficulties caused by the components a_j $(1 \leq j \leq n-1)$ on the boundary $\partial\mathbb{R}^n_+$, because no information on the boundary is known. Here the *tangential parity condition* is not required, however we have to pay the price, and treat this decay question on the local domain $\mathbb{R}^{n-1} \times (0, \sqrt{t}\,)$. Generally speaking, the Stokes solution does not exist in $L^1(\mathbb{R}^n_+)$ due to the counterexample given in [16]. However, Theorem 1.3 shows that the Stokes solution belongs to $L^1(\mathbb{R}^{n-1} \times (0, \sqrt{t}\,))$ under the assumption: the initial data a is in a weighted space. It should be pointed out that it is the first time for us to find the faster L^1-decay results of the arbitrary order spatial derivatives of the Stokes flow $e^{-tA}a$.

The large time decay for higher-order spatial derivatives of Navier–Stokes flows in $L^1(\mathbb{R}^n_+)$ is another unsolved problem. The main difficulties are that usual $L^q - L^r$ estimates for the Stokes flow fail in this case, and the projection operator $P:\ L^1(\mathbb{R}^n_+) \to L^1_\sigma(\mathbb{R}^n_+)$ becomes unbounded. Using the Stokes Solonnikov's solution formula, we decompose the operator P into two parts, and reduce its unboundedness to establish an L^1 estimate for an elliptic problem with Neumann boundary condition, which is overcome by using the weighted estimates of the Gaussian kernel's convolution. In fact, the presence of the boundary causes several difficulties in dealing with (1.1). In the case of the Cauchy problem, the projection P commutes with the Laplacian Δ; so the semigroup $\{e^{-tA}\}_{t\geq 0}$ is essentially equal to the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$, which is bounded on the L^1 space of solenoidal fields. Moreover, P can be written in terms of the Riesz transforms, and so one can avoid the use of $L^1(\mathbb{R}^n)$ by employing the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ in which P is bounded. However, all of these techniques are not applicable to problem (1.1) on the half-spaces $\mathbb{R}^n_+$, because the projection operator $P:\ L^1(\mathbb{R}^n_+) \to L^1_\sigma(\mathbb{R}^n_+)$ may be unbounded, which causes many difficulties in dealing with the L^1-decays of flows of (1.1). To our knowledge, few L^1-decay results are available for solutions of (1.1) including their first and second spatial derivatives.

Theorem 1.4. *Assume $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$ and*

$$\||x|u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+|x|)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+|x|)u_0\|_{L^1(\mathbb{R}^n_+)} < \infty,$$

then for $0 < \alpha < 1$, if u_0 is small in $L^n_\sigma(\mathbb{R}^n_+)$, the strong solution (u,p) given in Theorem 1.0 satisfies

$$\|u(t)\|_{L^1(\mathbb{R}^{(n-1)}\times(0,\sqrt{t}\,))} \leq C_\alpha t^{-\frac{\alpha}{2}}, \quad \forall\, t > 0,$$

$$\|\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \le C_\alpha t^{-\frac{1}{2}-\frac{\alpha}{2}}, \ \forall\, t > 0,$$
$$\|\nabla^2 u(t)\|_{L^1(\mathbb{R}^n_+)} \le C_\alpha t^{-1-\frac{\alpha}{2}}, \ \forall\, t > 1.$$

Remark. (1) Theorem 1.4 shows that if the initial data u_0 are in appropriate weighted spaces, the Navier–Stokes flows not only lie in $L^1(\mathbb{R}^{(n-1)} \times (0, \sqrt{t}\,))$, but also tend to 0 in $L^1(\mathbb{R}^{n-1} \times (0, \sqrt{t}\,))$ as $t \longrightarrow \infty$. There is no information on the asymptotic behavior at $t = 0$ for the estimates $\||x|^\alpha u(t)\|_{L^2(\mathbb{R}^n_+)}$, $\||x|^\alpha \nabla u(t)\|_{L^2(\mathbb{R}^n_+)}$ with $0 < \alpha < 1$, which are essential in the proof of Theorem 1.4. This is why we require the initial data u_0 to be in some weighted spaces.

(2) As pointed out in Remark (2) in Theorem 1.2 in [25], if the initial data are in some weighted spaces, it is an open question whether Theorem 1.4 holds without the *tangential parity condition*. In Theorem 1.4, we give a partial answer to this question. According to our knowledge, it is the first time to give the faster decay estimates: $\|\nabla^k u(t)\|_{L^1(\mathbb{R}^n_+)}$ with $k = 1, 2$.

The paper is organized as follows: In Section 2 we collect some basic and known results regarding the strong solution of (1.1), which will be applied in the subsequent sections. By means of the properties of the Gaussian kernel's convolution, we give some crucial weighted estimates of the convection term arising from (1.1), in fact, such a study is of independent interest. Together with the regularity theory of the steady Stokes system, we establish large time decays of the higher-order norms of flows of (1.1), see Theorem 1.1. Section 3 is devoted to the weighted decays of higher-order derivatives of the strong solution of (1.1). To do this, splitting the half-space into two parts, that is, $\mathbb{R}^n_+ = \big(\mathbb{R}^{n-1} \times (0, 1]\big) \cup \big(\mathbb{R}^{n-1} \times (1, \infty)\big)$. Using the conclusions obtained in Theorem 1.1 and Solonnikov's solution formula, we focus and work on the two sub-regions, respectively, then achieve the desired results, e.g. Theorem 1.2. In Section 4, we investigate the faster L^1-decay estimates of the Stokes flow (including its arbitrary order spatial derivatives), and then give a partial answer to L^1-behavior on time of the Navier–Stokes flows (including the first and second order spatial derivatives) in Section 5, this is a long standing problem.

2. Decay results of higher-order spatial derivatives

In this article, the integral $\| \int_{\mathbb{R}^n_+} K(\cdot, y) f(y) dy\|_{L^q(\mathbb{R}^n_+)}$ denotes

$$\Big\| \int\limits_{\mathbb{R}^n_+} K(\cdot, y) f(y) dy \Big\|_{L^r(\mathbb{R}^n_+)} = \begin{cases} \big(\int_{\mathbb{R}^n_+} \big| \int_{\mathbb{R}^n_+} K(x, y) f(y) dy \big|^r dx\big)^{\frac{1}{r}} & \text{if } 1 \le r < \infty, \\ ess \sup\limits_{x \in \mathbb{R}^n_+} \big| \int_{\mathbb{R}^n_+} K(x, y) f(y) dy \big| & \text{if } r = \infty. \end{cases}$$

In this article we frequently use the following inequalities (2.2), (2.3) and (2.4), which are variants of the classical Young inequality (see [39]). Let $K(x, y)$, $f(y)$ be defined in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, $\mathbb{R}^n_+$, respectively, and let $1 \le p, q \le r \le \infty$ satisfy $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then

$$\Big\| \int_{\mathbb{R}^n_+} K(\cdot,y)f(y)dy\Big\|_{L^r(\mathbb{R}^n_+)}$$
$$\le \sup_{x\in\mathbb{R}^n_+} \|K(x,\cdot)\|^{1-\frac{p}{r}}_{L^p(\mathbb{R}^n_+)} \sup_{y\in\mathbb{R}^n_+} \|K(\cdot,y)\|^{\frac{p}{r}}_{L^p(\mathbb{R}^n_+)} \|f\|_{L^q(\mathbb{R}^n_+)}. \tag{2.1}$$

Here we give a sketch of the proof of (2.1). Using Hölder inequality yields

$$\Big| \int_{\mathbb{R}^n_+} K(x,y)f(y)dy\Big| \le \|K(x,\cdot)\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_+)} \|f\|_{L^q(\mathbb{R}^n_+)},$$

and then

$$\Big\| \int_{\mathbb{R}^n_+} K(\cdot,y)f(y)dy\Big\|_{L^r(\mathbb{R}^n_+)} \le \Big\| \|K(x,\cdot)\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_+)}\Big\|_{L^r(\mathbb{R}^n_+)} \|f\|_{L^q(\mathbb{R}^n_+)}.$$

It is sufficient to prove

$$\Big\| \|K(x,\cdot)\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_+)}\Big\|_{L^r(\mathbb{R}^n_+)} \le \sup_{x\in\mathbb{R}^n_+} \|K(x,\cdot)\|^{1-\frac{p}{r}}_{L^p(\mathbb{R}^n_+)} \sup_{y\in\mathbb{R}^n_+} \|K(\cdot,y)\|^{\frac{p}{r}}_{L^p(\mathbb{R}^n_+)}.$$

In fact,

$$\begin{aligned}
&\Big\| \|K(x,\cdot)\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_+)}\Big\|^r_{L^r(\mathbb{R}^n_+)} \\
&= \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} |K(x,y)|^{\frac{q}{q-1}} dy\Big)^{\frac{(q-1)r}{q}} dx \\
&= \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} |K(x,y)|^{p+\frac{p^2}{r-p}} dy\Big)^{\frac{(q-1)r}{q}} dx \\
&\le \int_{\mathbb{R}^n_+} \sup_{y\in\mathbb{R}^n_+} |K(x,y)|^{\frac{(q-1)rp^2}{q(r-p)}} \Big(\int_{\mathbb{R}^n_+} |K(x,y)|^p dy\Big)^{\frac{(q-1)r}{q}} dx \\
&\le \sup_{x\in\mathbb{R}^n_+} \|K(x,\cdot)\|^{r-p}_{L^p(\mathbb{R}^n_+)} \sup_{y\in\mathbb{R}^n_+} \|K(\cdot,y)\|^p_{L^p(\mathbb{R}^n_+)},
\end{aligned}$$

where we use the simple facts: $p = \frac{(q-1)rp^2}{q(r-p)}$, $r-p = \frac{(q-1)rp}{q}$. Whence the inequality (2.1) holds.

It follows from (2.1) that

$$\Big\| \int_{\mathbb{R}^n_+} K(\cdot,y)f(y)dy\Big\|_{L^1(\mathbb{R}^n_+)} \le \sup_{y\in\mathbb{R}^n_+} \|K(\cdot,y)\|_{L^1(\mathbb{R}^n_+)} \|f\|_{L^1(\mathbb{R}^n_+)}; \tag{2.2}$$

and

$$\Big\| \int_{\mathbb{R}^n_+} K(\cdot, y) f(y) dy \Big\|_{L^r(\mathbb{R}^n_+)} \le \big(\sup_{x \in \mathbb{R}^n_+} \|K(x, \cdot)\|_{L^p(\mathbb{R}^n_+)} + \sup_{y \in \mathbb{R}^n_+} \|K(\cdot, y)\|_{L^p(\mathbb{R}^n_+)} \big) \|f\|_{L^q(\mathbb{R}^n_+)}. \tag{2.3}$$

On the other hand, checking the above proof process of (2.1), we find if $K(x,y) = K(x-y)$, $f(y)$ are defined in the whole space $\mathbb{R}^n$. Then

$$\Big\| \int_{\mathbb{R}^n} K(x-y) f(y) dy \Big\|_{L^r(\mathbb{R}^n)} \le \|K\|_{L^p(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)}. \tag{2.4}$$

In (2.3), (2.4), the numbers $1 \le p, q \le r \le \infty$ satisfy $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Lemma 2.1. *(See [19,22,23].) Let $u_0 \in L^1(\mathbb{R}^n_+) \bigcap L^q_\sigma(\mathbb{R}^n_+)$ $(n \ge 2)$ for all $1 < q < \infty$. If u_0 is small in $L^n_\sigma(\mathbb{R}^n_+)$, then the strong solution (u,p) given in Theorem 1.0 satisfies for $t > 1$*

$$\|\nabla^m u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-\frac{m}{2} - \frac{n}{2}(1-\frac{1}{r})}, \;\; m = 0,1,2, \quad 1 < r \le \infty;$$
$$\|\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla p(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-1-\frac{n}{2}(1-\frac{1}{r})}, \quad 1 < r < \infty.$$

Further, if u_0 satisfies $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, then

$$\|\nabla^m u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-\frac{m+1}{2} - \frac{n}{2}(1-\frac{1}{r})}, \;\; m = 0,1,2, \quad 1 < r \le \infty;$$

and

$$\|\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla p(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})}, \quad 1 < r < \infty.$$

In addition, if

$$\|(1+|x|)u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+|x|)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+|x|)u_0\|_{L^1(\mathbb{R}^n_+)} < \infty,$$

then it holds for $0 < \beta < 1$ and $t > 0$

$$\||x|^\beta u(t)\|_{L^2(\mathbb{R}^n_+)} + \||x|^\beta \nabla u(t)\|_{L^2(\mathbb{R}^n_+)} \le C_\beta (1+t)^{-\frac{n}{4}+\frac{\beta}{2}}.$$

Let $g = \mathcal{N} f$ denote the solution of the Neumann problem

$$-\Delta g = f \quad \text{in} \quad \mathbb{R}^n_+, \qquad \partial_\nu g\,|_{\partial \mathbb{R}^n_+} = 0.$$

Then (see [21])

$$\mathcal{N} = \int_0^\infty F(\tau) d\tau, \tag{2.5}$$

where the operator $F(t)$ is defined by

$$F(t)f(x) = \int\limits_{\mathbb{R}^n_+} [G_t(x'-y', x_n - y_n) + G_t(x'-y', x_n + y_n)]f(y)dy,$$

and $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel.

Moreover for $u, v \in L^2_\sigma(\mathbb{R}^n_+) \bigcap H^1_0(\mathbb{R}^n_+)$

$$P(u \cdot \nabla v) = u \cdot \nabla v + \sum_{i,j=1}^{n} \nabla \mathcal{N} \partial_i \partial_j (u_i v_j). \tag{2.6}$$

Lemma 2.2. *Let $m \geq 0$ be an integer, $0 \leq \alpha < 1$ and $1 \leq q \leq \infty$, $\frac{1}{q} = \frac{1}{q_i} + \frac{1}{q_i'}$, $q \leq q_i \leq \infty$, $i = 1, 2, 3, 4, 5$. Then for $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$*

$$\begin{aligned} \|x_n^\alpha \nabla^m P(u \cdot \nabla u)\|_{L^q(\mathbb{R}^n_+)} \leq C \sum_{\ell=0}^{[\frac{m}{2}]} \Big(&\|\nabla^\ell u\|_{L^{q_1}(\mathbb{R}^n_+)} \|\nabla^{m-\ell} u\|_{L^{q_1'}(\mathbb{R}^n_+)} \\ &+ \|\nabla^{\ell+1} u\|_{L^{q_2}(\mathbb{R}^n_+)} \|\nabla^{m+1-\ell} u\|_{L^{q_2'}(\mathbb{R}^n_+)} \\ &+ \|\nabla^\ell u\|_{L^{q_3}(\mathbb{R}^n_+)} \|y_n^\alpha \nabla^{m-\ell} u\|_{L^{q_3'}(\mathbb{R}^n_+)} \\ &+ \|\nabla^\ell u\|_{L^{q_4}(\mathbb{R}^n_+)} \|y_n^\alpha \nabla^{m+1-\ell} u\|_{L^{q_4'}(\mathbb{R}^n_+)} \\ &+ \|\nabla^{\ell+1} u\|_{L^{q_5}(\mathbb{R}^n_+)} \|y_n^\alpha \nabla^{m+1-\ell} u\|_{L^{q_5'}(\mathbb{R}^n_+)} \Big), \end{aligned} \tag{2.7}$$

where $[\frac{m}{2}]$ denotes the integer part of the number $\frac{m}{2}$.

Proof. Denote the odd and even extensions of a function f from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, respectively, by

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ -f(x', -x_n) & \text{if } x_n < 0, \end{cases} \quad \text{and} \quad f_*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ f(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Let $1 \leq k \leq n$ and $0 \leq \alpha < 1$. From (2.5), we get

$$\begin{aligned} &\| \sum_{i,j=1}^{n} x_n^\alpha \nabla^m \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \|_{L^q(\mathbb{R}^n_+)} \\ &= \| \sum_{i,j=1}^{n} x_n^\alpha \nabla^m \partial_k \int_0^\infty F(\tau) \partial_i \partial_j (u_i u_j) d\tau \|_{L^q(\mathbb{R}^n_+)} \\ &= \| \sum_{i,j=1}^{n} x_n^\alpha \nabla^m \partial_k \Big(\int_0^1 + \int_1^\infty \Big) G_\tau * [\partial_i \partial_j (u_i u_j)]_* d\tau \|_{L^q(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&\le C\Big\|\int_0^1\int_{\mathbb{R}^n}|x_n-y_n|^\alpha|\partial_k G_\tau(\cdot-y)||\nabla^m[\sum_{i,j=1}^n\partial_i\partial_j(u_iu_j)]_*(y)|dyd\tau\Big\|_{L^q(\mathbb{R}^n)}\\
&\quad+C\Big\|\int_0^1\int_{\mathbb{R}^n}|\partial_k G_\tau(\cdot-y)||y_n|^\alpha|\nabla^m[\sum_{i,j=1}^n\partial_i\partial_j(u_iu_j)]_*(y)|dyd\tau\Big\|_{L^q(\mathbb{R}^n)}\\
&\quad+C\sum_{i,j=1}^n\Big\|\int_1^\infty\int_{\mathbb{R}^n}|x_n-y_n|^\alpha|\partial_k\partial_i\partial_j G_\tau(\cdot-y)||\nabla^m\widetilde{w}_{ij}(y)|dyd\tau\Big\|_{L^q(\mathbb{R}^n)}\\
&\quad+C\sum_{i,j=1}^n\Big\|\int_1^\infty\int_{\mathbb{R}^n}|\partial_k\partial_i\partial_j G_\tau(\cdot-y)||y_n|^\alpha|\nabla^m\widetilde{w}_{ij}(y)|dyd\tau\Big\|_{L^q(\mathbb{R}^n)}\\
&=I_1+I_2+I_3+I_4, \qquad (2.8)
\end{aligned}
$$

where $\widetilde{w}_{ij}=(u_iu_j)_*$ if $1\le i,j\le n-1$ or $i=j=n$; $\widetilde{w}_{in}=(u_iu_n)^*$ if $1\le i\le n-1$; $\widetilde{w}_{nj}=(u_nu_j)^*$ if $1\le j\le n-1$.

$$
\begin{aligned}
I_1+I_2&\le C\int_0^1\||x_n|^\alpha\partial_kG_\tau\|_{L^1(\mathbb{R}^n)}d\tau\|\nabla^m(\nabla u\cdot\nabla u)\|_{L^q(\mathbb{R}^n_+)}\\
&\quad+C\int_0^1\|\partial_kG_\tau\|_{L^1(\mathbb{R}^n)}d\tau\|y_n^\alpha\nabla^m(\nabla u\cdot\nabla u)\|_{L^q(\mathbb{R}^n_+)}\\
&\le C\||x_n|^\alpha\partial_kG_1\|_{L^1(\mathbb{R}^n)}\int_0^1\tau^{\frac{\alpha}{2}-\frac12}d\tau\|\nabla^m(\nabla u\cdot\nabla u)\|_{L^q(\mathbb{R}^n_+)}\\
&\quad+C\|\partial_kG_1\|_{L^1(\mathbb{R}^n)}\int_0^1\tau^{-\frac12}d\tau\|y_n^\alpha\nabla^m(\nabla u\cdot\nabla u)\|_{L^q(\mathbb{R}^n_+)}\\
&\le C\sum_{\ell=0}^{[\frac{m}{2}]}\big(\|\nabla^{\ell+1}u\|_{L^{q_2}(\mathbb{R}^n_+)}\|\nabla^{m+1-\ell}u\|_{L^{q_2'}(\mathbb{R}^n_+)}\\
&\quad+\|\nabla^{\ell+1}u\|_{L^{q_5}(\mathbb{R}^n_+)}\|y_n^\alpha\nabla^{m+1-\ell}u\|_{L^{q_5'}(\mathbb{R}^n_+)}\big); \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
I_3&\le C\sum_{i,j=1}^n\int_1^\infty\int_{\mathbb{R}^n}|x_n|^\alpha|\partial_k\partial_i\partial_jG_\tau(x',x_n)|dxd\tau\|\nabla^m\widetilde{w}_{ij}\|_{L^q(\mathbb{R}^n_+)}\\
&\le C\int_1^\infty\tau^{\frac{\alpha}{2}-\frac32}d\tau\sum_{i,j=1}^n\int_{\mathbb{R}^n}|x_n|^\alpha|\partial_k\partial_i\partial_jG_1(x',x_n)|dx
\end{aligned}
$$

$$\times \sum_{\ell=0}^{[\frac{m}{2}]} \|\nabla^\ell u\|_{L^{q_1}(\mathbb{R}^n_+)} \|\nabla^{m-\ell} u\|_{L^{q_1'}(\mathbb{R}^n_+)}$$

$$\leq C \sum_{\ell=0}^{[\frac{m}{2}]} \|\nabla^\ell u\|_{L^{q_1}(\mathbb{R}^n_+)} \|\nabla^{m-\ell} u\|_{L^{q_1'}(\mathbb{R}^n_+)}; \tag{2.10}$$

$$I_4 \leq C \sum_{i,j=1}^{n} \int_1^\infty \tau^{-\frac{3}{2}} d\tau \|\partial_i \partial_j \partial_k G_1\|_{L^1(\mathbb{R}^n)} \|y_n^\alpha \nabla^m \widetilde{w}_{ij}\|_{L^q(\mathbb{R}^n_+)}$$

$$\leq C \sum_{\ell=0}^{[\frac{m}{2}]} \|\nabla^\ell u\|_{L^{q_3}(\mathbb{R}^n_+)} \|y_n^\alpha \nabla^{m-\ell} u\|_{L^{q_3'}(\mathbb{R}^n_+)}. \tag{2.11}$$

On the other hand,

$$\|x_n^\alpha \nabla^m (u \cdot \nabla u)\|_{L^q(\mathbb{R}^n_+)} \leq C \sum_{\ell=0}^{[\frac{m}{2}]} \|\nabla^\ell u\|_{L^{q_4}(\mathbb{R}^n_+)} \|x_n^\alpha \nabla^{m+1-\ell} u\|_{L^{q_4'}(\mathbb{R}^n_+)}. \tag{2.12}$$

From (2.6), (2.8)–(2.12), we infer that (2.7) holds. □

Proof of Theorem 1.1. Set $v(t) = \partial_t u(t)$. Then it follows from problem (1.1) that $v(t)$ satisfies

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla u + u \cdot \nabla v) + \nabla \partial_t p = 0 & \text{in } \mathbb{R}^n_+ \times (1, \infty), \\ \nabla \cdot v = 0 & \text{in } \mathbb{R}^n_+ \times (1, \infty), \\ v(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (1, \infty), \\ v(x,1) = \partial_t u(1) & \text{in } \mathbb{R}^n_+. \end{cases} \tag{2.13}$$

Moreover, it holds for $t > 2$ (see [40,41])

$$v(x,t) = \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,\frac{t}{2}) v(y,\frac{t}{2}) dy - \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s) P(v \cdot \nabla u + u \cdot \nabla v)(y,s) dy ds, \tag{2.14}$$

where $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$ is defined as follows for $1 \leq i,j \leq n$

$$M_{ij}(x,y,t) = \delta_{ij}(G_t(x-y) - G_t(x-y^*)) + M^*_{ij}(x,y,t) = \delta_{ij} G_t(x-y) + N^*_{ij}(x,y,t).$$

Here

$$M^*_{ij}(x,y,t) = 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_i} G_t(z-y^*)dz,$$

and

$$N^*_{ij}(x,y,t) = -\delta_{ij}G_t(x-y^*) + M^*_{ij}(x,y,t)$$

$y^* = (y_1, y_2, \cdots, -y_n)$, $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel, and

$$E(z) = \begin{cases} -\frac{\Gamma(\frac{n}{2})}{2(n-2)\pi^{\frac{n}{2}}}\frac{1}{|z|^{n-2}} & \text{if } n > 2, \\ \frac{1}{2\pi}\log|z| & \text{if } n = 2, \end{cases}$$

is the fundamental solution of the Laplace equation. In addition, the following estimate holds for $M^* = (M^*_{ij})_{n\times n}$, $N^* = (N^*_{ij})_{n\times n}$:

$$\begin{aligned} &|\partial_t^{\widetilde{s}}\partial_x^{\widetilde{\ell}}\partial_y^{\widetilde{m}} M^*(x,y,t)| + |\partial_t^{\widetilde{s}}\partial_x^{\widetilde{\ell}}\partial_y^{\widetilde{m}} N^*(x,y,t)| \\ &\le Ct^{-s-\frac{\widetilde{m}_n}{2}}(t+x_n^2)^{-\frac{\widetilde{\ell}_n}{2}}(|x-y^*|^2+t)^{-\frac{n+|\widetilde{\ell}'|+|\widetilde{m}'|}{2}} e^{-\frac{cy_n^2}{t}}, \end{aligned} \tag{2.15}$$

where $\widetilde{m} = (\widetilde{m}_1, \widetilde{m}_2, \cdots, \widetilde{m}_{n-1}, \widetilde{m}_n) = (\widetilde{m}', \widetilde{m}_n)$, $\widetilde{\ell} = (\widetilde{\ell}_1, \widetilde{\ell}_2, \cdots, \widetilde{\ell}_{n-1}, \widetilde{\ell}_n) = (\widetilde{\ell}', \widetilde{\ell}_n)$.

Let $1 < r < \infty$, $n \ge 2$. Using (2.3), (2.4), (2.15) and Lemma 2.1, we find for each integer $m \ge 1$ and $t > 2$

$$\begin{aligned} &\big\|\nabla_x^m \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, \frac{t}{2})v(y,\frac{t}{2})dy\big\|_{L^r(\mathbb{R}^n_+)} \\ &\le C\|\nabla^m G_t\|_{L^1(\mathbb{R}^n_+)}\|v(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} + C\|v(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} \times \\ &\quad \sum_{m=m_n+|m'|}\int_{\mathbb{R}^n_+}(x_n+\sqrt{t})^{-m_n}\big(|x'|+x_n+\sqrt{t}\big)^{-n-|m'|}dx'dx_n \\ &\le Ct^{-\frac{m+2}{2}-\frac{n}{2}(1-\frac{1}{r})}; \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x \mathcal{M}(\cdot, y, t-s)P(v\cdot\nabla u + u\cdot\nabla v)(y,s)dyds\big\|_{L^r(\mathbb{R}^n_+)} \\ &\le C\int_{\frac{t}{2}}^{t} \|\nabla G_{t-s}\|_{L^1(\mathbb{R}^n_+)}\|P(v\cdot\nabla u + u\cdot\nabla v)(s)\|_{L^r(\mathbb{R}^n_+)}ds \end{aligned}$$

$$+ C \int_{\frac{t}{2}}^{t} \big(\sup_{x \in \mathbb{R}^n_+} \|\nabla_x \mathcal{N}^*(x, \cdot, t-s)\|_{L^1(\mathbb{R}^n_+)} + \sup_{y \in \mathbb{R}^n_+} \|\nabla_x \mathcal{N}^*(\cdot, y, t-s)\|_{L^1(\mathbb{R}^n_+)} \big)$$

$$\times \|P(v \cdot \nabla u + u \cdot \nabla v)(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \|P\|_{\mathcal{L}(L^r \to L^r_\sigma)} \big(\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|u(s)\|_{L^\infty(\mathbb{R}^n_+)} \|\nabla v(s)\|_{L^r(\mathbb{R}^n_+)} \big) ds$$

$$\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-1-\frac{n}{2}(1-\frac{1}{2r})-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{2r})} ds$$

$$+ C f_1(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{n}{2}-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})} ds$$

$$\leq C t^{-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + C_1 t^{-\frac{n-1}{2}-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})} f_1(t), \tag{2.17}$$

where $f_1(t) = \sup\limits_{0<s\leq t} [s^{\frac{3}{2}+\frac{n}{2}(1-\frac{1}{r})} \|\nabla v(s)\|_{L^r(\mathbb{R}^n_+)}]$.

Combining (2.14), (2.16) with (2.17), we conclude for $t > 2$

$$t^{\frac{3}{2}+\frac{n}{2}(1-\frac{1}{r})} \|\nabla v(t)\|_{L^r(\mathbb{R}^n_+)} \leq C + C_1 t^{-\frac{n-1}{2}} f_1(t),$$

which implies that for $t > 2$

$$f_1(t) \leq C + C_1 t^{-\frac{n-1}{2}} f_1(t). \tag{2.18}$$

There exists $t_1 > 2$ such that $C_1 t_1^{-\frac{n-1}{2}} \leq \frac{1}{2}$. Then (2.18) yields for $t \geq t_1$, $f_1(t) \leq 2C$, from which we get for $1 < r < \infty$ and $t \geq t_1$

$$\|\nabla \partial_t u(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla v(t)\|_{L^r(\mathbb{R}^n_+)} \leq C t^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.19}$$

Now we recall the regularity estimates of the steady Stokes equations in half-spaces (see p. 197, Lemma 3.2; p. 198, Theorem 3.1 in Section 3, Chapter IV in [20]).

Let $f, g \in C_0^\infty(\overline{\mathbb{R}^n_+})$, $m \geq 0$, $q \in (1, \infty)$. Then any solution (w, π) to the nonhomogeneous Stokes system

$$\begin{cases} -\Delta w + \nabla \pi = f & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot w = g & \text{in } \mathbb{R}^n_+, \\ w(x) = 0 & \text{on } \partial \mathbb{R}^n_+, \\ w(x) \to 0 & \text{as } x \in \mathbb{R}^n_+, \ |x| \to \infty, \end{cases} \tag{2.20}$$

is C^∞, and belongs to $W^{m+2,q}(\mathbb{R}^n_+) \times W^{m+1,q}(\mathbb{R}^n_+)$. Moreover, for each integer $\ell \geq 0$, the following estimate holds

$$\|\nabla^{\ell+2} w\|_{L^q(\mathbb{R}^n_+)} + \|\nabla^{\ell+1}\pi\|_{L^q(\mathbb{R}^n_+)} \leq c(n,\ell,q)(\|\nabla^\ell f\|_{L^q(\mathbb{R}^n_+)} + \|\nabla^{\ell+1} g\|_{L^q(\mathbb{R}^n_+)}). \tag{2.21}$$

Applying (2.21) to problem (1.1), we get for $1 < r < \infty$, $\ell \geq 0$ and $t > 0$

$$\begin{aligned}&\|\nabla^{\ell+2} u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^{\ell+1} p(t)\|_{L^r(\mathbb{R}^n_+)}\\ &\quad\leq C(\|\nabla^\ell(u \cdot \nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^\ell \partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}).\end{aligned} \tag{2.22}$$

Using Lemma 2.1, we find for $1 < r < \infty$ and $t \geq t_1$

$$\begin{aligned}\|\nabla(u \cdot \nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} &\leq C(\|\nabla u(t)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2 u(t)\|_{L^{2r}(\mathbb{R}^n_+)})\\ &\leq Ct^{-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}.\end{aligned} \tag{2.23}$$

Using (2.19), (2.22) and (2.23), we obtain for $1 < r < \infty$ and $t \geq t_1$

$$\begin{aligned}&\|\nabla^3 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^2 p(t)\|_{L^r(\mathbb{R}^n_+)}\\ &\leq C(\|\nabla(u \cdot \nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla \partial_t u(t)\|_{L^r(\mathbb{R}^n_+)})\\ &\leq Ct^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})}.\end{aligned} \tag{2.24}$$

Note that for $0 < s < t$ and $x = (x', x_n) \in \mathbb{R}^n_+$,

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+} \partial_{x_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))g(y,s)dy\\ &= -\int\limits_{\mathbb{R}^n_+} \partial_{y_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))g(y,s)dy\\ &\quad - 2\int\limits_{\mathbb{R}^n_+} \partial_{x_n} G_{t-s}(x-y^*)g(y,s)dy\\ &= \int\limits_{\mathbb{R}^n_+} (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n} g(y,s)dy\\ &\quad - 2\int\limits_{\mathbb{R}^n_+} \partial_{x_n} G_{t-s}(x-y^*)g(y,s)dy,\end{aligned} \tag{2.25}$$

which yields for $1 \leq k \leq n$ and $0 < s < t$

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_n} G_{t-s}(x-y)g(y,s)dy \\
&= \int_{\mathbb{R}^n_+} \partial_{x_k}(G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n} g(y,s)dy \\
&\quad - \int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_n} G_{t-s}(x-y^*)g(y,s)dy. \qquad (2.26)
\end{aligned}$$

Let $1 < r < \infty$. Using (2.3), (2.4), (2.15), (2.19), (2.26) and Lemmata 2.1, 2.2, we find for $1 \le k \le n$, $1 \le j \le n-1$ and $t \ge 2t_1$

$$\begin{aligned}
&\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_j}\mathcal{M}(\cdot,y,t-s)P(v\cdot\nabla u + u\cdot\nabla v)(y,s)dyds\|_{L^r(\mathbb{R}^n_+)} \\
&+ \| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_n}\mathcal{M}(\cdot,y,t-s)P(v\cdot\nabla u + u\cdot\nabla v)(y,s)dyds\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\int_{\frac{t}{2}}^{t} \|\partial_{x_k} G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n_+)}(\|\partial_n P(v\cdot\nabla u + u\cdot\nabla v)(s)\|_{L^r(\mathbb{R}^n_+)} \\
&+ \|\partial_j P(v\cdot\nabla u + u\cdot\nabla v)(s)\|_{L^r(\mathbb{R}^n_+)})ds \\
&+ C\int_{\frac{t}{2}}^{t} \|x_n^{2\epsilon}\partial_{x_k}\partial_{x_n} G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n_+)}\|y_n^{-2\epsilon} P(v\cdot\nabla u + u\cdot\nabla v)(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&+ C\int_{\frac{t}{2}}^{t} \Big(\sup_{x=(x',x_n)\in\mathbb{R}^n_+} \|(x_n+y_n)^{2\epsilon}\partial_{x_k}\partial_{x_n}\mathcal{N}^*(x,\cdot,t-s)\|_{L^1(\mathbb{R}^n_+)} \\
&+ \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \|(x_n+y_n)^{2\epsilon}\partial_{x_k}\partial_{x_n}\mathcal{N}^*(\cdot,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\Big) \\
&\times \|z_n^{-2\epsilon} P(v\cdot\nabla u + u\cdot\nabla v)(\cdot,s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&+ C\int_{\frac{t}{2}}^{t} \Big(\sup_{x\in\mathbb{R}^n_+} \|\partial_{x_k}\mathcal{N}^*(x,\cdot,t-s)\|_{L^1(\mathbb{R}^n_+)} + \sup_{y\in\mathbb{R}^n_+} \|\partial_{x_k}\mathcal{N}^*(\cdot,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\Big) \\
&\times \|\partial_j P(v\cdot\nabla u + u\cdot\nabla v)(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&\le C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}}(\|u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)}
\end{aligned}$$

$$
\begin{aligned}
&+\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\\
&+\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^{2r}(\mathbb{R}^n_+)}+\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^{2r}(\mathbb{R}^n_+)})ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12}(\|u(s)\|_{L^\infty(\mathbb{R}^n_+)}+\|\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)})\|\nabla^2 v(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}(\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\\
&+\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}+\|u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\\
&+\|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)})ds\\
\le{}& C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12}s^{-\frac32-\frac n2-\frac n2(1-\frac1r)}ds+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}s^{-1-\frac n2-\frac n2(1-\frac1r)}ds\\
&+Cf_2(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12}s^{-\frac n2}X_0(s)ds\\
\le{}& C_\epsilon t^{\epsilon-1-\frac n2-\frac n2(1-\frac1r)}+Ct^{-\frac{n-1}{2}}X_0(t)f_2(t),
\end{aligned}
\tag{2.27}
$$

where $\epsilon\in(0,\frac12)$, $f_2(t)=\sup\limits_{0<s\le t}[X_0(s^{-1})\|\nabla^2 v(s)\|_{L^r(\mathbb{R}^n_+)}]$, and

$$
X_0(t)=\begin{cases} t^{-2-\frac n2(1-\frac1r)} & \text{if } n\ge 3,\\ t^{\epsilon-2-(1-\frac1r)} & \text{if } n=2.\end{cases}
$$

Using (2.14), (2.16) and (2.27), we get for $\epsilon\in(0,\frac12)$ and $t\ge 2t_1$

$$
X_0(t^{-1})\|\nabla^2 v(t)\|_{L^r(\mathbb{R}^n_+)}\le C+C_2t^{-\frac{n-1}{2}}f_2(t). \tag{2.28}
$$

Note that there exists $t_2\ge 2t_1$ such that $C_2t_2^{-\frac{n-1}{2}}\le\frac12$. Whence (2.28) yields for $\epsilon\in(0,\frac12)$ and $t\ge t_2$

$$
f_2(t)\le C+\frac12 f_2(t),\quad \text{and then}\quad f_2(t)\le 2C,
$$

which implies for $1<r<\infty$ and $t\ge t_2$

$$
\|\nabla^2\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}=\|\nabla^2 v(t)\|_{L^r(\mathbb{R}^n_+)}\le\begin{cases} Ct^{-2-\frac n2(1-\frac1r)} & \text{if } n\ge 3,\\ C_\epsilon t^{\epsilon-2-(1-\frac1r)} & \text{if } n=2,\end{cases} \tag{2.29}
$$

where $\epsilon\in(0,\frac12)$.

Combining (2.22), (2.24), (2.29) and Lemma 2.1, we find for $1 < r < \infty$ and $t \geq t_2$

$$\begin{aligned} &\|\nabla^4 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^3 p(t)\|_{L^r(\mathbb{R}^n_+)} \\ &\leq C(\|\nabla^2(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^2\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}) \\ &\leq \begin{cases} Ct^{-2-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \geq 3, \\ C_\epsilon t^{\epsilon-2-(1-\frac{1}{r})} & \text{if } n = 2, \end{cases} \quad \text{where} \quad \epsilon \in (0, \frac{1}{2}). \end{aligned} \tag{2.30}$$

In order to get the decay of $\|\nabla^5 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4 p(t)\|_{L^r(\mathbb{R}^n_+)}$, using the regularity estimate (2.22) for the steady Stokes system, we find for $t > 1$

$$\|\nabla^5 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4 p(t)\|_{L^r(\mathbb{R}^n_+)} \leq C(\|\nabla^3(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^3\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}). \tag{2.31}$$

Using (2.14) and (2.15), we find that the strong singularity appears in estimating $\|\nabla^3 v(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla^3\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}$, which becomes too difficult for us to treat by using the method employed in the proof of (2.30). To avoid such singularity, we apply the regularity estimate (see (2.21)) to problem (2.13), and get for $t > 1$

$$\|\nabla^3 v(t)\|_{L^r(\mathbb{R}^n_+)} \leq C(\|\nabla(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}). \tag{2.32}$$

So it is sufficient to establish the decay of $\|\nabla\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}$ in (2.32). Indeed, using (2.31) and (2.32), we can obtain the desired estimate of $\|\nabla^5 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4 p(t)\|_{L^r(\mathbb{R}^n_+)}$. To do this, setting $\widetilde{v}(t) = \partial_t v(t)$, then from (2.13), we find for $t > 1$

$$\begin{cases} \partial_t\widetilde{v} - \Delta\widetilde{v} + (\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v}) + \nabla\partial_{tt}p = 0 & \text{in } \mathbb{R}^n_+ \times (1,\infty), \\ \nabla\cdot\widetilde{v} = 0 & \text{in } \mathbb{R}^n_+ \times (1,\infty), \\ \widetilde{v}(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (1,\infty), \\ \widetilde{v}(x,1) = \partial_{tt}u(1) & \text{in } \mathbb{R}^n_+. \end{cases} \tag{2.33}$$

Furthermore, it holds for $t > 2$ (see [40,41])

$$\begin{aligned} \widetilde{v}(x,t) = &\int_{\mathbb{R}^n_+} \mathcal{M}(x,y,\frac{t}{2})\widetilde{v}(y,\frac{t}{2})dy \\ &- \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s)P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(y,s)dyds. \end{aligned} \tag{2.34}$$

Note that for each $1 < r < \infty$, the projection $P: L^r(\mathbb{R}^n_+) \longrightarrow L^r_\sigma(\mathbb{R}^n_+)$ is bounded. Whence, using (2.19), (2.29) and Lemma 2.1, we get for $t \geq t_2$

$$
\begin{aligned}
&\|\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)} = \|\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)} \\
&\le \|Av(t)\|_{L^r(\mathbb{R}^n_+)} + \|P(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\|P\|_{\mathcal{L}(L^r\to L^r_\sigma)}(\|\nabla^2 v(t)\|_{L^r(\mathbb{R}^n_+)} + \|v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(t)\|_{L^{2r}(\mathbb{R}^n_+)}) \\
&\le Ct^{-\frac{3}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + \begin{cases} Ct^{-2-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3, \\ C_\epsilon t^{\epsilon-2-(1-\frac{1}{r})} & \text{if } n=2, \end{cases} \\
&\le \begin{cases} Ct^{-2-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3, \\ C_\epsilon t^{\epsilon-2-(1-\frac{1}{r})} & \text{if } n=2, \end{cases} \quad \text{where} \quad \epsilon\in(0,\frac{1}{2}).
\end{aligned} \tag{2.35}
$$

Let $1<r<\infty$ and $m\ge 1$ be an integer. Using (2.2)–(2.4), (2.15), (2.19), (2.35) and Lemma 2.1, we find for $\epsilon\in(0,\frac{1}{2})$ and $t\ge 2t_2$

$$
\begin{aligned}
&\|\nabla_x^m \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot,y,\frac{t}{2})\widetilde{v}(y,\frac{t}{2})dy\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\|\nabla^m G_t\|_{L^1(\mathbb{R}^n_+)}\|\widetilde{v}(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} + C\|\widetilde{v}(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} \\
&\quad\times \sum_{|m'|+m_n=m}\int_{\mathbb{R}^n_+}(x_n+\sqrt{t}\,)^{-m_n}\big(|x'|+x_n+\sqrt{t}\,\big)^{-n-|m'|}dx'dx_n \\
&\le \begin{cases} Ct^{-2-\frac{m}{2}-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3, \\ C_\epsilon t^{\epsilon-2-\frac{m}{2}-(1-\frac{1}{r})} & \text{if } n=2, \end{cases}
\end{aligned} \tag{2.36}
$$

and

$$
\begin{aligned}
&\|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\nabla_x\mathcal{M}(\cdot,y,t-s)P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(y,s)dyds\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\int_{\frac{t}{2}}^{t}\|\nabla G_{t-s}\|_{L^1(\mathbb{R}^n_+)}\|P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&\quad + C\int_{\frac{t}{2}}^{t}\big(\sup_{x\in\mathbb{R}^n_+}\|\nabla_x\mathcal{N}^*(x,\cdot,t-s)\|_{L^1(\mathbb{R}^n_+)} + \sup_{y\in\mathbb{R}^n_+}\|\nabla_x\mathcal{N}^*(\cdot,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\big) \\
&\quad \times\|P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(\|\widetilde{v}(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}
\end{aligned}
$$

$$+\|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\big)ds$$
$$+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\|u(s)\|_{L^\infty(\mathbb{R}^n_+)}\|\nabla\widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}ds$$
$$\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}}X_1(s)ds+Cf_3(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}}X_1(s)ds$$
$$\le Ct^{-\frac{n-1}{2}}X_1(t)+Ct^{-\frac{n-1}{2}}X_1(t)f_3(t), \tag{2.37}$$

where $f_3(t)=\sup\limits_{0<s\le t}[X_1(s^{-1})\|\nabla\widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}]$, and

$$X_1(t)=\begin{cases} t^{-\frac{5}{2}-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3,\\ t^{\epsilon-\frac{5}{2}-(1-\frac{1}{r})} & \text{if } n=2.\end{cases}$$

Combining (2.34), (2.36) with (2.37), we conclude for $\epsilon\in(0,\frac{1}{2})$ and $t\ge 2t_2$

$$X_1(t^{-1})\|\nabla\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)}\le C+C_3t^{-\frac{n-1}{2}}f_3(t). \tag{2.38}$$

There is $t_3\ge 2t_2$ such that $C_3t_3^{-\frac{n-1}{2}}\le\frac{1}{2}$. Whence (2.38) yields for $\epsilon\in(0,\frac{1}{2})$ and $t\ge t_3$, $f_3(t)\le C+\frac{1}{2}f_3(t)$, and then $f_3(t)\le 2C$. This shows for $1<r<\infty$, $\epsilon\in(0,\frac{1}{2})$ and $t\ge t_3$

$$\|\nabla\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}=\|\nabla\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)}\le\begin{cases} Ct^{-\frac{5}{2}-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3,\\ C_\epsilon t^{\epsilon-\frac{5}{2}-(1-\frac{1}{r})} & \text{if } n=2.\end{cases} \tag{2.39}$$

From (2.29), (2.31), (2.32), (2.39) and Lemma 2.1, we conclude for $1<r<\infty$ and $t\ge t_3$

$$\|\nabla^5u(t)\|_{L^r(\mathbb{R}^n_+)}+\|\nabla^4p(t)\|_{L^r(\mathbb{R}^n_+)}$$
$$\le C(\|\nabla^3(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)}+\|\nabla^3v(t)\|_{L^r(\mathbb{R}^n_+)})$$
$$\le C(\|\nabla^2u(t)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^3u(t)\|_{L^{2r}(\mathbb{R}^n_+)}$$
$$+\|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^4u(t)\|_{L^{2r}(\mathbb{R}^n_+)})$$
$$+C(\|\nabla v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)}+\|v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^{2r}(\mathbb{R}^n_+)}$$
$$+\|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2v(t)\|_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla\partial_tv(t)\|_{L^r(\mathbb{R}^n_+)})$$
$$\le\begin{cases} Ct^{-\frac{5}{2}-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n\ge 3,\\ C_\epsilon t^{\epsilon-\frac{5}{2}-(1-\frac{1}{r})} & \text{if } n=2,\end{cases}\quad\text{where}\quad \epsilon\in(0,\frac{1}{2}). \tag{2.40}$$

Now we try to get the decay of $\|\nabla^6u(t)\|_{L^r(\mathbb{R}^n_+)}+\|\nabla^5p(t)\|_{L^r(\mathbb{R}^n_+)}$. Using the regularity theory (see (2.22)), we find for $t>1$

$$\|\nabla^6 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^5 p(t)\|_{L^r(\mathbb{R}^n_+)} \le C(\|\nabla^4(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}). \tag{2.41}$$

Applying the regularity estimate (2.21) to problem (2.13), we get for $t > 1$

$$\begin{aligned}\|\nabla^4\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)} &= \|\nabla^4 v(t)\|_{L^r(\mathbb{R}^n_+)}\\ &\le C(\|\nabla^2(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^2\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}).\end{aligned} \tag{2.42}$$

So in order to obtain the estimate of $\|\nabla^6 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^5 p(t)\|_{L^r(\mathbb{R}^n_+)}$, it is sufficient to give the decay of $\|\nabla^2\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}$ by means of (2.41) and (2.42).

Let $1 < r < \infty$. Using (2.15), (2.19), (2.26), (2.29), (2.35), (2.39), Lemmata 2.1, 2.2, we find for $1 \le k \le n$, $1 \le j \le n-1$ and $t \ge 2t_3$

$$\begin{aligned}
&\|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_k}\partial_{x_j}\mathcal{M}(\cdot,y,t-s)P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(y,s)dyds\|_{L^r(\mathbb{R}^n_+)}\\
&+\|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_k}\partial_{x_n}\mathcal{M}(\cdot,y,t-s)P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(y,s)dyds\|_{L^r(\mathbb{R}^n_+)}\\
\le{}& C\int_{\frac{t}{2}}^{t}\|\partial_k G_{t-s}\|_{L^1(\mathbb{R}^n_+)}(\|\partial_n P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)\|_{L^r(\mathbb{R}^n_+)}\\
&+\|\partial_j P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)\|_{L^r(\mathbb{R}^n_+)})ds\\
&+C\int_{\frac{t}{2}}^{t}\|x_n^{2\epsilon}\partial_{x_k}\partial_{x_n}G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n_+)}\\
&\times\|z_n^{-2\epsilon}P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(\cdot,s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&+C\int_{\frac{t}{2}}^{t}\Big(\sup_{(x',x_n)\in\mathbb{R}^n_+}\|(x_n+y_n)^{2\epsilon}\partial_{x_k}\partial_{x_n}\mathcal{N}^*(x,\cdot,t-s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\sup_{(y',y_n)\in\mathbb{R}^n_+}\|(x_n+y_n)^{2\epsilon}\partial_{x_k}\partial_{x_n}\mathcal{N}^*(\cdot,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\Big)\\
&\times\|z_n^{-2\epsilon}P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(\cdot,s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&+C\int_{\frac{t}{2}}^{t}\Big(\sup_{x\in\mathbb{R}^n_+}\|\partial_{x_k}\mathcal{N}^*(x,\cdot,t-s)\|_{L^1(\mathbb{R}^n_+)} + \sup_{y\in\mathbb{R}^n_+}\|\partial_{x_k}\mathcal{N}^*(\cdot,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \|\partial_j P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\Big[\|\widetilde{v}(s)\|_{L^{2r}(\mathbb{R}^n_+)}\sum_{\ell=1}^{3}\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \|\nabla\widetilde{v}(s)\|_{L^{2r}(\mathbb{R}^n_+)}\sum_{\ell=0}^{3}\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \sum_{i=0}^{1}\sum_{j=1}^{2}\|\nabla^i v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^j v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\Big]ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\sum_{\ell=0}^{2}\|\nabla^\ell u(s)\|_{L^\infty(\mathbb{R}^n_+)}\|\nabla^2\widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}\Big[\sum_{\ell=0}^{1}\|\nabla^\ell v(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|v(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla v(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \sum_{i=0}^{1}\sum_{j=0}^{1}\|\nabla^i\widetilde{v}(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^j u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\Big]ds \\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n-1}{2}}X_2(s)ds + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}s^{-2-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}ds \\
&\quad + Cf_4(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}}X_2(s)ds \\
&\le C_\epsilon t^{\epsilon-2-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + Ct^{-\frac{n-1}{2}}X_2(t)f_4(t),
\end{aligned}
\tag{2.43}
$$

where $\epsilon \in (0, \frac{1}{2})$, $f_4(t) = \sup_{0<s\le t}[X_2(s^{-1})\|\nabla^2\widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}]$, and

$$
X_2(t) = \begin{cases} t^{-3-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \ge 3, \\ t^{\epsilon-3-(1-\frac{1}{r})} & \text{if } n = 2. \end{cases}
$$

Combining (2.34), (2.36) and (2.43), we conclude for $\epsilon \in (0, \frac{1}{2})$ and $t \ge 2t_3$

$$
X_2(t^{-1})\|\nabla^2\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)} \le C + C_4 t^{-\frac{n-1}{2}}f_4(t). \tag{2.44}
$$

There exists $t_4 \ge 2t_3$ such that $C_4 t_4^{-\frac{n-1}{2}} \le \frac{1}{2}$. Whence (2.44) yields for $t \ge t_4$, $f_4(t) \le C + \frac{1}{2}f_4(t)$, and so $f_4(t) \le 2C$. This implies for $1 < r < \infty$ and $t \ge t_4$

$$\|\nabla^2\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla^2\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)} \le \begin{cases} Ct^{-3-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \ge 3, \\ C_\epsilon t^{\epsilon-3-(1-\frac{1}{r})} & \text{if } n = 2. \end{cases} \tag{2.45}$$

Applying (2.21) to problem (2.13), we get for $1 < r < \infty$ and $t > 1$

$$\|\nabla^4 v(t)\|_{L^r(\mathbb{R}^n_+)} \le C(\|\nabla^2(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^2\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}). \tag{2.46}$$

From (2.41), (2.42), (2.45) and (2.46), we conclude for $1 < r < \infty$, $\epsilon \in (0, \frac{1}{2})$ and $t \ge t_4$

$$\begin{aligned} &\|\nabla^6 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^5 p(t)\|_{L^r(\mathbb{R}^n_+)} \\ &\le C(\|\nabla^4(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4 v(t)\|_{L^r(\mathbb{R}^n_+)}) \\ &\le C(\|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^5 u(t)\|_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^4 u(t)\|_{L^{2r}(\mathbb{R}^n_+)} \\ &\quad + \|\nabla^2 u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^3 u(t)\|_{L^{2r}(\mathbb{R}^n_+)}) + C(\|\nabla^2 v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)} \\ &\quad + \|\nabla v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2 u(t)\|_{L^{2r}(\mathbb{R}^n_+)} + \|v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^3 u(t)\|_{L^{2r}(\mathbb{R}^n_+)} \\ &\quad + \|\nabla^3 v(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}) + \begin{cases} Ct^{-3-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \ge 3, \\ C_\epsilon t^{\epsilon-3-(1-\frac{1}{r})} & \text{if } n = 2 \end{cases} \\ &\le \begin{cases} Ct^{-3-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \ge 3, \\ C_\epsilon t^{\epsilon-3-(1-\frac{1}{r})} & \text{if } n = 2. \end{cases} \end{aligned} \tag{2.47}$$

Repeating the proofs of (2.24), (2.30), (2.40) and (2.47), it is not difficult to prove that for every integer $k \ge 1$, there exists $t_k > 0$ independent of t, such that for $1 < r < \infty$, $\epsilon \in (0, \frac{1}{2})$ and $t \ge t_k$

$$\|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^{1+k} p(t)\|_{L^r(\mathbb{R}^n_+)} \le \begin{cases} Ct^{-\frac{2+k}{2}-\frac{n}{2}(1-\frac{1}{r})} & \text{if } n \ge 3, \\ C_\epsilon t^{\epsilon-\frac{2+k}{2}-(1-\frac{1}{r})} & \text{if } n = 2. \end{cases} \tag{2.48}$$

Here the constants $C = C(k, r, n, u_0)$ and $C_\epsilon = C(\epsilon, k, r, n, u_0)$.

Recall the Gagliardo–Nirenberg inequality on the half-space (see (4.1) in [10]): Let $f \in W^{1,s}(\mathbb{R}^n_+)$, $n < s < \infty$. Then

$$\|h\|_{L^\infty(\mathbb{R}^n_+)} \le C\|h\|^{1-\frac{n}{s}}_{L^s(\mathbb{R}^n_+)}\|\nabla h\|^{\frac{n}{s}}_{L^s(\mathbb{R}^n_+)}. \tag{2.49}$$

Therefore, using (2.49) we conclude for every integer $k \ge 1$ and $t \ge t_k$,

$$\|\nabla^{2+k} u(t)\|_{L^\infty(\mathbb{R}^n_+)} \le C\|\nabla^{2+k} u(t)\|^{\frac{1}{2}}_{L^{2n}(\mathbb{R}^n_+)}\|\nabla^{3+k} u(t)\|^{\frac{1}{2}}_{L^{2n}(\mathbb{R}^n_+)};$$

and

$$\|\nabla^{1+k}p(t)\|_{L^\infty(\mathbb{R}^n_+)} \leq C\|\nabla^{1+k}p(t)\|^{\frac{1}{2}}_{L^{2n}(\mathbb{R}^n_+)}\|\nabla^{2+k}p(t)\|^{\frac{1}{2}}_{L^{2n}(\mathbb{R}^n_+)},$$

from which (2.48) is true for $r = \infty$. The proof of the estimate (1.2) is complete.

In the next arguments, we suppose $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, $n \geq 2$.

Using the second part of Lemma 2.1, following the proofs of (2.16), (2.17), we get for each integer $m \geq 1$ and $t > 2$

$$\|\nabla^m_x \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, \frac{t}{2})v(y, \frac{t}{2})dy\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}-\frac{m+2}{2}-\frac{n}{2}(1-\frac{1}{r})}; \tag{2.16$'$}$$

and

$$\begin{aligned}
&\|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x\mathcal{M}(\cdot, y, t-s)P(v\cdot\nabla u + u\cdot\nabla v)(y, s)dyds\|_{L^r(\mathbb{R}^n_+)} \\
&\leq C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-2-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}ds + C\widetilde{f}_1(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n+1}{2}-2-\frac{n}{2}(1-\frac{1}{r})}ds \\
&\leq Ct^{-2-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + Ct^{-\frac{n}{2}-2-\frac{n}{2}(1-\frac{1}{r})}\widetilde{f}_1(t),
\end{aligned} \tag{2.17$'$}$$

where $\widetilde{f}_1(t) = \sup_{0<s\leq t}[s^{2+\frac{n}{2}(1-\frac{1}{r})}\|\nabla v(s)\|_{L^r(\mathbb{R}^n_+)}]$.

Using (2.16′), (2.17′), following the proof of (2.19), we readily get for $1 < r < \infty$ and $t \geq \widehat{t}_1$ with some number $\widehat{t}_1 > 2$

$$\|\nabla\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla v(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-2-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.19$'$}$$

Using Lemma 2.1, we find for $1 < r < \infty$ and $t \geq \widehat{t}_1$

$$\begin{aligned}
\|\nabla(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} &\leq C(\|\nabla u(t)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|u(t)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^2 u(t)\|_{L^{2r}(\mathbb{R}^n_+)}) \\
&\leq Ct^{-2-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}.
\end{aligned} \tag{2.23$'$}$$

By means of (2.22), (2.19′) and (2.23′), we obtain for $1 < r < \infty$ and $t \geq \widehat{t}_1$

$$\begin{aligned}
&\|\nabla^3 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^2 p(t)\|_{L^r(\mathbb{R}^n_+)} \\
&\leq C(\|\nabla(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}) \\
&\leq Ct^{-2-\frac{n}{2}(1-\frac{1}{r})}.
\end{aligned} \tag{2.24$'$}$$

Using Lemmata 2.1, 2.2, following the proof of (2.27), we get for $1 \leq k \leq n$, $1 \leq j \leq n-1$, $1 < r < \infty$ and $t \geq 2\widehat{t}_1$

$$
\begin{aligned}
&\Big\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_j}\mathcal{M}(\cdot,y,t-s)P(v\cdot\nabla u+u\cdot\nabla v)(y,s)dyds\Big\|_{L^r(\mathbb{R}^n_+)} \\
&+\Big\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_n}\mathcal{M}(\cdot,y,t-s)P(v\cdot\nabla u+u\cdot\nabla v)(y,s)dyds\Big\|_{L^r(\mathbb{R}^n_+)} \\
\le\ & C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12}s^{-\frac52-\frac n2-\frac n2(1-\frac1r)}ds + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}s^{-2-\frac n2-\frac n2(1-\frac1r)}ds \\
&+C\widetilde{f}_2(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac12}s^{-\frac{n+1}{2}-\frac52-\frac n2(1-\frac1r)}ds \\
\le\ & C_\epsilon t^{\epsilon-2-\frac n2-\frac n2(1-\frac1r)}+Ct^{-\frac n2-\frac52-\frac n2(1-\frac1r)}\widetilde{f}_2(t),
\end{aligned}
\tag{2.27$'$}
$$

where $\widetilde{f}_2(t)=\sup\limits_{0<s\le t}[s^{\frac52+\frac n2(1-\frac1r)}\|\nabla^2 v(s)\|_{L^r(\mathbb{R}^n_+)}]$, $\epsilon\in(0,\frac12)$.

Repeating the proof's process of (2.29), using (2.16′), (2.17′), we get for $1<r<\infty$ and $t\ge\widehat{t}_2$ with some number $\widehat{t}_2\ge 2\widehat{t}_1$

$$
\|\nabla^2\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}=\|\nabla^2 v(t)\|_{L^r(\mathbb{R}^n_+)}\le Ct^{-\frac52-\frac n2(1-\frac1r)}. \tag{2.29$'$}
$$

By (2.22), (2.24′), (2.29′) and Lemma 2.1, we find for $1<r<\infty$ and $t\ge\widehat{t}_2$

$$
\begin{aligned}
&\|\nabla^4 u(t)\|_{L^r(\mathbb{R}^n_+)}+\|\nabla^3 p(t)\|_{L^r(\mathbb{R}^n_+)} \\
\le\ & C(\|\nabla^2(u\cdot\nabla u)(t)\|_{L^r(\mathbb{R}^n_+)}+\|\nabla^2\partial_t u(t)\|_{L^r(\mathbb{R}^n_+)}) \\
\le\ & Ct^{-\frac52-\frac n2(1-\frac1r)}.
\end{aligned}
\tag{2.30$'$}
$$

Utilizing Lemma 2.1, checking the proofs of (2.35)–(2.37), we get for $1<r<\infty$ and $t\ge\widehat{t}_3$ with $\widehat{t}_3\ge 2\widehat{t}_2$

$$
\|\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)}=\|\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)}\le Ct^{-\frac52-\frac n2(1-\frac1r)}; \tag{2.35$'$}
$$

and for each integer $m\ge1$,

$$
\Big\|\nabla_x^m\int_{\mathbb{R}^n_+}\mathcal{M}(\cdot,y,\frac t2)\widetilde{v}(y,\frac t2)dy\Big\|_{L^r(\mathbb{R}^n_+)}\le Ct^{-\frac52-\frac m2-\frac n2(1-\frac1r)}; \tag{2.36$'$}
$$

$$\begin{aligned}
&\Big\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \nabla_x \mathcal{M}(\cdot, y, t-s) P(\widetilde{v} \cdot \nabla u + 2v \cdot \nabla v + u \cdot \nabla \widetilde{v})(y, s) dy ds \Big\|_{L^r(\mathbb{R}^n_+)} \\
&\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{7}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} ds + C \widetilde{f}_3(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{n+1}{2}-3-\frac{n}{2}(1-\frac{1}{r})} ds \\
&\leq C t^{-3-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + C t^{-\frac{n}{2}-3-\frac{n}{2}(1-\frac{1}{r})} \widetilde{f}_3(t),
\end{aligned} \tag{2.37$'$}$$

where $\widetilde{f}_3(t) = \sup_{0<s\leq t} [s^{3+\frac{n}{2}(1-\frac{1}{r})} \|\nabla \widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}]$.

Following the proof procedure of (2.39), we have for $1 < r < \infty$ and $t \geq \widehat{t}_3$

$$\|\nabla \partial_t v(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla \widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)} \leq C t^{-3-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.39$'$}$$

From (2.31), (2.32), (2.29′) and (2.39′), we conclude for $1 < r < \infty$ and $t \geq \widehat{t}_3$

$$\|\nabla^5 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^4 p(t)\|_{L^r(\mathbb{R}^n_+)} \leq C t^{-3-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.40$'$}$$

Using Lemma 2.1, following the proof process of (2.43), we get for $1 \leq k \leq n$, $1 \leq j \leq n-1$, $1 < r < \infty$ and $t \geq \widehat{t}_3$

$$\begin{aligned}
&\Big\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k} \partial_{x_j} \mathcal{M}(\cdot, y, t-s) P(\widetilde{v} \cdot \nabla u + 2v \cdot \nabla v + u \cdot \nabla \widetilde{v})(y, s) dy ds \Big\|_{L^r(\mathbb{R}^n_+)} \\
&+ \Big\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_k} \partial_{x_n} \mathcal{M}(\cdot, y, t-s) P(\widetilde{v} \cdot \nabla u + 2v \cdot \nabla v + u \cdot \nabla \widetilde{v})(y, s) dy ds \Big\|_{L^r(\mathbb{R}^n_+)} \\
&\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{7}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} ds + C \int_{\frac{t}{2}}^{t} (t-s)^{-1+\epsilon} s^{-3-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} ds \\
&+ C \widetilde{f}_4(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{n+1}{2}-\frac{7}{2}-\frac{n}{2}(1-\frac{1}{r})} ds \\
&\leq C_\epsilon t^{\epsilon-3-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + C t^{-\frac{n}{2}-\frac{7}{2}-\frac{n}{2}(1-\frac{1}{r})} \widetilde{f}_4(t),
\end{aligned} \tag{2.43$'$}$$

where $\widetilde{f}_4(t) = \sup_{0<s\leq t} [s^{\frac{7}{2}+\frac{n}{2}(1-\frac{1}{r})} \|\nabla^2 \widetilde{v}(s)\|_{L^r(\mathbb{R}^n_+)}]$.

Checking the proof of (2.45), together with (2.43′), we infer that for $1 < r < \infty$ and $t \geq \widehat{t}_3$

$$\|\nabla^2\partial_t v(t)\|_{L^r(\mathbb{R}^n_+)} = \|\nabla^2\widetilde{v}(t)\|_{L^r(\mathbb{R}^n_+)} \le Ct^{-\frac{7}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.45'}$$

Utilizing (2.45′) and following the proof procedure of (2.47) yield for $t \ge \widehat{t}_4$ with $\widehat{t}_4 \ge 2\widehat{t}_3$

$$\|\nabla^6 u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^5 p(t)\|_{L^r(\mathbb{R}^n_+)} \le Ct^{-\frac{7}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.47'}$$

Repeating the proofs of (2.24′), (2.30′), (2.40′) and (2.47′), we can find for every integer $k \ge 1$, there exist $\widehat{C}_k > 0$ and $\widehat{t}_k > 0$ such that for $1 < r < \infty$ and $t \ge \widehat{t}_k$

$$\|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^{1+k} p(t)\|_{L^r(\mathbb{R}^n_+)} \le \widehat{C}_k t^{-\frac{1}{2}-\frac{2+k}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{2.48'}$$

Using the Gagliardo–Nirenberg inequality (2.49), we infer for every integer $k \ge 1$ and $t \ge \widehat{t}_k$, (2.48′) is also valid for $r = \infty$. The proof of (1.3) is complete. □

3. Weighted decay properties of higher-order norms

This section is devoted to the weighted decays of $\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)}$, where $0 < \beta < 1$, $1 < r \le \infty$, and u is the strong solution of (1.1), which is given in Theorem 1.0. Using the Solonnikov's solution formula (see [40,41]), we have for $t > 0$

$$u(x,t) = \int\limits_{\mathbb{R}^n_+} \mathcal{M}(x,y,\frac{t}{2})u(y,\frac{t}{2})dy - \int\limits_{\frac{t}{2}}^{t}\int\limits_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s)P(u\cdot\nabla u)(y,s)dyds. \tag{3.1}$$

To avoid the strong singularity in estimating $\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)}$, we split the half-space into two parts, that is, $\mathbb{R}^n_+ = \big(\mathbb{R}^{n-1} \times (0,1]\big) \cup \big(\mathbb{R}^{n-1} \times (1,\infty)\big)$. Then we deal with the weighted decays of higher-order derivatives in the two sub-regions, respectively. Exactly speaking, the first decay part of $\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1}\times(0,1])}$ can be treated by means of the conclusions obtained in Theorem 1.1, and the second estimate of $\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))}$ can be solved by making full use of the special structure in the explicit Solonnikov's solution formula.

Proof of Theorem 1.2. Note that for $1 \le \ell \le n-1$ and $t > 0$

$$\begin{aligned}&\int\limits_{\mathbb{R}^n_+} \partial_{x_\ell}(G_{\frac{t}{2}}(x-y) - G_{\frac{t}{2}}(x-y^*))u(y,\frac{t}{2})dy \\ &\quad = \int\limits_{\mathbb{R}^n_+} (G_{\frac{t}{2}}(x-y) - G_{\frac{t}{2}}(x-y^*))\partial_{y_\ell} u(y,\frac{t}{2})dy,\end{aligned} \tag{3.2}$$

and

$$\int_{\mathbb{R}^n_+} \partial_{x_n}(G_{\frac{t}{2}}(x-y) - G_{\frac{t}{2}}(x-y^*))u(y,\frac{t}{2})dy$$
$$= \int_{\mathbb{R}^n_+} (G_{\frac{t}{2}}(x-y) + G_{\frac{t}{2}}(x-y^*))\partial_{y_n}u(y,\frac{t}{2})dy. \tag{3.3}$$

Let $0 < \beta < 1$, $1 < r \le \infty$, and let $k \ge 1$, $k_1, k_2 \ge 0$ such that $2 + k = k_1 + k_2$. Then using (2.15), (3.2), (3.3) and Lemma 2.1 yields for $t > 2$

$$\begin{aligned}
&\|x_n^\beta \nabla_x^{2+k} \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, \frac{t}{2})u(y,\frac{t}{2})dy\|_{L^r(\mathbb{R}^n_+)} \\
&\le \|x_n^\beta \int_{\mathbb{R}^n_+} (|\nabla_x^{1+k} G_{\frac{t}{2}}(\cdot - y)| + |\nabla_x^{1+k} G_{\frac{t}{2}}(\cdot - y^*)|)|\nabla u(y,\frac{t}{2})dy\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + \|x_n^\beta \int_{\mathbb{R}^n_+} \nabla_x^{2+k}\mathcal{M}^*(\cdot, y, \frac{t}{2})u(y,\frac{t}{2})dy\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\| \int_{\mathbb{R}^n_+} |x_n - y_n|^\beta |\nabla_x^{1+k} G_{\frac{t}{2}}(\cdot - y)||\nabla u(y,\frac{t}{2})|dy\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + C\| \int_{\mathbb{R}^n_+} |\nabla_x^{1+k} G_{\frac{t}{2}}(\cdot - y)|y_n^\beta |\nabla u(y,\frac{t}{2})|dy\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + C\| \int_{\mathbb{R}^n_+} (x_n + y_n)^\beta |\nabla_x^{1+k} G_{\frac{t}{2}}(\cdot - y^*)||\nabla u(y,\frac{t}{2})|dy\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + C\| \int_{\mathbb{R}^n_+} (x_n + y_n)^\beta (\frac{t}{2} + x_n^2)^{-\frac{k_1}{2}}(|\cdot - y^*|^2 + \frac{t}{2})^{-\frac{n+k_2}{2}}|u(y,\frac{t}{2})|dy\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\|x_n^\beta \nabla^{1+k} G_{\frac{t}{2}}\|_{L^1(\mathbb{R}^n_+)}\|\nabla u(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} + C\|\nabla^{1+k} G_{\frac{t}{2}}\|_{L^1(\mathbb{R}^n_+)}\|y_n^\beta \nabla u(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + C\|u(\frac{t}{2})\|_{L^r(\mathbb{R}^n_+)} \int_{\mathbb{R}^n_+} x_n^\beta (\sqrt{t} + x_n)^{-k_1}(|x'| + x_n + \sqrt{t})^{-n-k_2} dx' dx_n \\
&\le C t^{\frac{\beta}{2} - \frac{2+k}{2} - \frac{n}{2}(1 - \frac{1}{r})},
\end{aligned} \tag{3.4}$$

where we used the known result (see Theorem 1.2 in [24]): Under the assumptions of Theorem 1.2, it holds for $0 < \beta < 1$ and $t > 1$

$$\|y_n^\beta \nabla u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{\frac{\beta}{2} - \frac{1}{2} - \frac{n}{2}(1 - \frac{1}{r})}, \quad 1 < r \le \infty$$

for $n \ge 2$ with the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$; and $n \ge 3$ without such assumption.

Using (2.25), we find for $0 < s < t$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \partial^2_{x_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))f(y,s)dy \\
&= \int\limits_{\mathbb{R}^n_+} \partial_{x_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n} f(y,s)dy \\
&\quad - 2\int\limits_{\mathbb{R}^n_+} \partial^2_{x_n} G_{t-s}(x-y^*)f(y,s)dy \\
&= \int\limits_{\mathbb{R}^n_+} (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial^2_{y_n} f(y,s)dy \\
&\quad - 2\int\limits_{\mathbb{R}^n_+} \partial_{x_n} G_{t-s}(x-y^*)\partial_{y_n} f(y,s)dy \\
&\quad - 2\int\limits_{\mathbb{R}^n_+} \partial^2_{x_n} G_{t-s}(x-y^*)f(y,s)dy.
\end{aligned} \tag{3.5}$$

Now we claim that it holds for each integer $m \geq 1$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \partial^m_{x_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))f(y,s)dy \\
&= \int\limits_{\mathbb{R}^n_+} (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial^m_{y_n} f(y,s)dy \\
&\quad - 2\int\limits_{\mathbb{R}^n_+} \sum_{\ell=1}^{m} \partial^\ell_{x_n} G_{t-s}(x-y^*)\partial^{m-\ell}_{y_n} f(y,s)dy.
\end{aligned} \tag{3.6}$$

Indeed, from (2.25) and (3.5), we know that (3.6) holds true for $m = 1, 2$.

Suppose that (3.6) is true for given number $m \geq 2$. Then for $0 < s < t$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \partial^{m+1}_{x_n}(G_{t-s}(x-y) - G_{t-s}(x-y^*))f(y,s)dy \\
&= \int\limits_{\mathbb{R}^n_+} \partial^m_{x_n}(-\partial_{y_n})(G_{t-s}(x-y) + G_{t-s}(x-y^*))f(y,s)dy \\
&= \int\limits_{\mathbb{R}^n_+} \partial^m_{x_n}(-\partial_{y_n})(G_{t-s}(x-y) - G_{t-s}(x-y^*))f(y,s)dy \\
&\quad - 2\int\limits_{\mathbb{R}^n_+} \partial^{m+1}_{x_n} G_{t-s}(x-y^*)f(y,s)dy
\end{aligned}$$

$$= \int_{\mathbb{R}^n_+} \partial_{x_n}^m (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n} f(y,s)dy$$

$$- 2\int_{\mathbb{R}^n_+} \partial_{x_n}^{m+1} G_{t-s}(x-y^*) f(y,s)dy$$

$$= \int_{\mathbb{R}^n_+} (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n}^m \partial_{y_n} f(y,s)dy$$

$$- 2\int_{\mathbb{R}^n_+} \sum_{\ell=1}^{m} \partial_{x_n}^{\ell} G_{t-s}(x-y^*)\partial_{y_n}^{m-\ell}\partial_{y_n} f(y,s)dy$$

$$- 2\int_{\mathbb{R}^n_+} \partial_{x_n}^{m+1} G_{t-s}(x-y^*) f(y,s)dy$$

$$= \int_{\mathbb{R}^n_+} (G_{t-s}(x-y) - G_{t-s}(x-y^*))\partial_{y_n}^{m+1} f(y,s)dy$$

$$- 2\int_{\mathbb{R}^n_+} \sum_{\ell=1}^{m+1} \partial_{x_n}^{\ell} G_{t-s}(x-y^*)\partial_{y_n}^{m+1-\ell} f(y,s)dy.$$

The above arguments show that (3.6) is valid for all integer $m \geq 1$.

Let $1 < r \leq \infty$ and $k \geq 1$. Using (1.2) with $n \geq 3$, and by (2.3), (2.4), (2.15), (3.6) and Lemmata 2.1, 2.2, we obtain for $t \geq t_k$

$$\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_n}^{2+k} \mathcal{M}(\cdot, y, t-s) P(u\cdot\nabla u)(y,s)dyds\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))}$$

$$\leq C\int_{\frac{t}{2}}^{t} \||x_n|^\beta \partial_{x_n} G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n)} \|\partial_{y_n}^{1+k} Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C\int_{\frac{t}{2}}^{t} \|\partial_n G_{t-s}\|_{L^1(\mathbb{R}^n_+)} \|y_n^\beta \partial_{y_n}^{1+k} Pu(s)\cdot\nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C\sum_{\ell=1}^{1+k} \int_{\frac{t}{2}}^{t} \big(\sup_{x\in\mathbb{R}^n_+} \|(x_n+y_n)^\beta \partial_{x_n}^{1+\ell} G_{t-s}(x-y^*)\|_{L^1(\mathbb{R}^{n-1}\times(1,\infty))}$$

$$+ \sup_{y\in\mathbb{R}^n_+} \|(x_n+y_n)^\beta \partial_{x_n}^{1+\ell} G_{t-s}(\cdot - y^*)\|_{L^1(\mathbb{R}^{n-1}\times(1,\infty))}\big)$$

$$\times \|\partial_{y_n}^{1+k-\ell} Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C \int_{\frac{t}{2}}^{t} \big(\sup_{x \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \partial_{x_n}^{2+k} \mathcal{M}^*(x, \cdot, t-s)\|_{L^1(\mathbb{R}^{n-1} \times (1,\infty))}$$

$$+ \sup_{y \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \partial_{x_n}^{2+k} \mathcal{M}^*(\cdot, y, t-s)\|_{L^1(\mathbb{R}^{n-1} \times (1,\infty))} \big)$$

$$\times \|Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}+\frac{\beta}{2}} \|\partial_{y_n}^{1+k} Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \|y_n^\beta \partial_{y_n}^{1+k} Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C \int_{\frac{t}{2}}^{t} (t-s)^{-1+\frac{\beta}{2}} \|\partial_{y_n}^{k} Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds$$

$$+ C \int_{\frac{t}{2}}^{t} (t-s)^{-1+\epsilon} (\|Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)}$$

$$+ \|\partial_{y_n}^{k} Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)}) ds$$

$$\leq C \sum_{\ell=0}^{1+k} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}+\frac{\beta}{2}} \big(\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell+1} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \big) ds$$

$$+ C \sum_{\ell=0}^{k} \int_{\frac{t}{2}}^{t} (t-s)^{-1+\frac{\beta}{2}} (\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell+1} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}) ds$$

$$+ C \sum_{\ell=0}^{k} \int_{\frac{t}{2}}^{t} (t-s)^{-1+\epsilon} (\|u(s)\|_{L^{2r}(\mathbb{R}^n_+)}^2 + \|\nabla u(s)\|_{L^{2r}(\mathbb{R}^n_+)}^2$$

$$+ \|\nabla^{\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell+1} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}$$

$$+ \|\nabla^{\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)}) ds$$

$$
\begin{aligned}
&+C\sum_{\ell=0}^{1+k}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(\|\nabla^{\ell}u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^{1+k-\ell}u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\\
&+\|\nabla^{\ell+1}u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\|\nabla^{2+k-\ell}u(s)\|_{L^{2r}(\mathbb{R}^n_+)}\big)ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\Big(\sum_{\ell=0}^{1+k}\|\nabla^{\ell}u(s)\|_{L^{\infty}(\mathbb{R}^n_+)}\Big)\Big(\sum_{\ell=0}^{1+[\frac{1+k}{2}]}\|y_n^{\beta}\nabla^{2+k-\ell}u(s)\|_{L^{r}(\mathbb{R}^n_+)}\Big)ds\\
\le{}& C\sum_{\ell=0}^{1+k}\int_{\frac{t}{2}}^{t}((t-s)^{-\frac{1}{2}+\frac{\beta}{2}}+(t-s)^{-\frac{1}{2}})s^{-\frac{\ell}{2}-\frac{1+k-\ell}{2}-n(1-\frac{1}{2r})}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}s^{-\frac{1+k}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}ds+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}s^{-n(1-\frac{1}{2r})}ds\\
&+C\sum_{\ell=0}^{1+[\frac{1+k}{2}]}g_{2+k-\ell}(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}}Q_{r,2+k-\ell}(s)ds\\
\le{}& Ct^{\epsilon-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}+Ct^{-\frac{n-1}{2}}\sum_{\ell=\ell_0}^{2+k}Q_{r,\ell}(t)g_{\ell}(t),
\end{aligned}
\tag{3.7}
$$

where $n\ge 3$, $\epsilon\in(0,\frac{1}{2})$, $\ell_0=2+k-(1+[\frac{1+k}{2}])=1+k-[\frac{1+k}{2}]$,

$$
g_{\ell}(t)=\sup_{0<s\le t}[Q_{r,\ell}(s^{-1})\|y_n^{\beta}\nabla^{\ell}u(\cdot,s)\|_{L^{r}(\mathbb{R}^n_+)}],
$$

$$
Q_{r,\ell}(t)=t^{-\min\{\frac{\Theta(n,k,\epsilon)}{2},\frac{\ell}{2}-\frac{\beta}{2}\}-\frac{n}{2}(1-\frac{1}{r})},\quad \Theta(n,k,\epsilon)=\begin{cases}n-1 & \text{if } k=1,\\ n-2\epsilon & \text{if } k\ge 2.\end{cases}
$$

Observe that for $t>0$ and $1\le j\le n-1$, $\partial_{x_j}\mathcal{M}(x,y,t)=-\partial_{y_j}\mathcal{M}(x,y,t)$. Whence for $m\ge 1$, $1\le j\le n-1$ and $t\ge t_k$

$$
\begin{aligned}
&\partial_{x_j}^{m}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\mathcal{M}(x,y,t-s)P(u\cdot\nabla u)(y,s)dyds\\
={}&\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\Big((-1)^m\partial_{y_j}^{m}\mathcal{M}(x,y,t-s)\Big)P(u\cdot\nabla u)(y,s)dyds\\
={}&\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\mathcal{M}(x,y,t-s)\partial_{y_j}^{m}P(u\cdot\nabla u)(y,s)dyds.
\end{aligned}
\tag{3.8}
$$

Set $1 < r \le \infty$, $k \ge 1$ and $x = (x', x_n), y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1_+$. Using (1.2) with $n \ge 3$, (3.8), Lemmata 2.1, 2.2, we have for $t \ge t_k$

$$
\begin{aligned}
&\Big\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \nabla_{x'}^{2+k} \mathcal{M}(\cdot, y, t-s) P(u \cdot \nabla u)(y, s) dy ds\Big\|_{L^r(\mathbb{R}^n_+)} \\
&= \Big\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \nabla_{x'} \mathcal{M}(\cdot, y, t-s) \nabla_{y'}^{1+k} P(u \cdot \nabla u)(y, s) dy ds\Big\|_{L^r(\mathbb{R}^n_+)} \\
&\le C \int_{\frac{t}{2}}^{t} \|x_n^\beta \nabla_{x'} G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n_+)} \|\nabla_{y'}^{1+k} P u(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&\quad + C \int_{\frac{t}{2}}^{t} \|\nabla_{x'} G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n_+)} \|y_n^\beta \nabla_{y'}^{1+k} P u(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&\quad + C \int_{\frac{t}{2}}^{t} \Big(\sup_{x \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \nabla_{x'} \mathcal{N}^*(x, \cdot, t-s)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \sup_{y \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \nabla_{x'} \mathcal{N}^*(\cdot, y, t-s)\|_{L^1(\mathbb{R}^n_+)} \Big) \\
&\quad \times \|\nabla_{y'}^{1+k} P u(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&\le C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}+\frac{\beta}{2}} \|\nabla_{y'}^{1+k} P u(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&\quad + C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \|y_n^\beta \nabla_{y'}^{1+k} P u(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&\le C \sum_{\ell=0}^{1+k} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}+\frac{\beta}{2}} \big(\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \|\nabla^{\ell+1} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \big) ds \\
&\quad + C \sum_{\ell=0}^{1+k} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \big(\|\nabla^\ell u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{1+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \\
&\quad + \|\nabla^{\ell+1} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla^{2+k-\ell} u(s)\|_{L^{2r}(\mathbb{R}^n_+)} \big) ds
\end{aligned}
$$

$$+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\Big(\sum_{\ell=0}^{1+k}\|\nabla^{\ell}u(s)\|_{L^{\infty}(\mathbb{R}^n_+)}\Big)\Big(\sum_{\ell=0}^{1+[\frac{1+k}{2}]}\|y_n^{\beta}\nabla^{2+k-\ell}u(\cdot,s)\|_{L^r(\mathbb{R}^n_+)}\Big)ds$$

$$\leq C\sum_{\ell=0}^{1+k}\int_{\frac{t}{2}}^{t}((t-s)^{-\frac{1}{2}+\frac{\beta}{2}}+(t-s)^{-\frac{1}{2}})s^{-\frac{\ell}{2}-\frac{1+k-\ell}{2}-n(1-\frac{1}{2r})}ds$$

$$+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}s^{-n(1-\frac{1}{2r})}ds+C\sum_{\ell=0}^{1+[\frac{1+k}{2}]}g_{2+k-\ell}(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{n}{2}}Q_{r,\ell}(s)ds$$

$$\leq Ct^{\epsilon-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}+Ct^{-\frac{n-1}{2}}\sum_{\ell=\ell_0}^{2+k}Q_{r,\ell}(t)g_{\ell}(t), \tag{3.9}$$

where $n\geq 3$, $\epsilon\in(0,\frac{1}{2})$, and the definitions of $Q_{r,\ell}(t)$, $g_{\ell}(t)$ $(\ell=\ell_0,\ell_0+1,\cdots,2+k)$ have been given in the proof of (3.7).

Let $2+k=k_1+k_2$. Note that the cases of $\nabla_x^{2+k}=\nabla_{x'}^{k_1}\nabla_{x_n}^{k_2}$ with $k_1=0$, $k_2=0$ have been handled in (3.7), (3.9), respectively. So we assume $k_1\geq 1$ and $k_2\geq 1$. Let $1<r<\infty$ and $x=(x',x_n),y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^1_+$. Following the proofs of (3.7) and (3.9), we have for $n\geq 3$ and $t\geq t_k$

$$\Big\|x_n^{\beta}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\nabla_x^{2+k}\mathcal{M}(\cdot,y,t-s)P(u\cdot\nabla u)(y,s)dyds\Big\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))}$$

$$=\Big\|x_n^{\beta}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_n}^{k_2}\mathcal{M}(\cdot,y,t-s)\nabla_{y'}^{k_1}P(u\cdot\nabla u)(y,s)dyds\Big\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))}$$

$$\leq C\int_{\frac{t}{2}}^{t}\||x_n|^{\beta}\partial_{x_n}G_{t-s}(\cdot)\|_{L^1(\mathbb{R}^n)}\|\partial_{y_n}^{k_2-1}\nabla_{y'}^{k_1}Pu(s)\cdot\nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)}ds$$

$$+C\int_{\frac{t}{2}}^{t}\|\partial_n G_{t-s}\|_{L^1(\mathbb{R}^n_+)}\|y_n^{\beta}\partial_{y_n}^{k_2-1}\nabla_{y'}^{k_1}Pu(s)\cdot\nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)}ds$$

$$+C\sum_{\ell=0}^{k_2-1}\int_{\frac{t}{2}}^{t}\Big(\sup_{x\in\mathbb{R}^n_+}\|(x_n+y_n)^{\beta}\partial_{x_n}^{1+\ell}G_{t-s}(x-y^*)\|_{L^1(\mathbb{R}^{n-1}\times(1,\infty))}$$

$$+\sup_{y\in\mathbb{R}^n_+}\|(x_n+y_n)^{\beta}\partial_{x_n}^{1+\ell}G_{t-s}(\cdot-y^*)\|_{L^1(\mathbb{R}^{n-1}\times(1,\infty))}\Big)$$

$$\times\|\partial_{y_n}^{k_2-1-\ell}\nabla_{y'}^{k_1}Pu(s)\cdot\nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)}ds$$

$$
\begin{aligned}
&+ C \int_{\frac{t}{2}}^{t} \Big(\sup_{x \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \nabla_x^{2+k} \mathcal{M}^*(x, \cdot, t-s)\|_{L^1(\mathbb{R}^{n-1} \times (1,\infty))} \\
&+ \sup_{y \in \mathbb{R}^n_+} \|(x_n + y_n)^\beta \nabla_x^{2+k} \mathcal{M}^*(\cdot, y, t-s)\|_{L^1(\mathbb{R}^{n-1} \times (1,\infty))} \Big) \\
&\times \|Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} ds \\
\le\ & C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}+\frac{\beta}{2}} \|\nabla_y^{1+k} Pu(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&+ C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \|y_n^\beta \nabla_y^{1+k} Pu(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&+ C \int_{\frac{t}{2}}^{t} (t-s)^{-1+\frac{\beta}{2}} \|\partial_{y_n}^{k_2-1} \nabla_{y'}^{k_1} Pu(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} ds \\
&+ C \int_{\frac{t}{2}}^{t} (t-s)^{-1+\epsilon} (\|Pu(s) \cdot \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} \\
&+ \|\nabla_{y'}^{k_1} Pu(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)} + \|\nabla_{y'}^{k} Pu(s) \cdot \nabla u(s)(\cdot)\|_{L^r(\mathbb{R}^n_+)}) ds \\
\le\ & C t^{\epsilon - \frac{n}{2} - \frac{n}{2}(1-\frac{1}{r})} + C t^{-\frac{n-1}{2}} \sum_{\ell=\ell_0}^{2+k} Q_{r,\ell}(t) g_\ell(t),
\end{aligned} \tag{3.10}
$$

where $\epsilon \in (0, \frac{1}{2})$, and $Q_{r,\ell}(t)$, $g_\ell(t)$ ($\ell = \ell_0, \ell_0 + 1, \cdots, 2+k$) arise from the proof of (3.7).

From (3.1), (3.4), (3.7), (3.9) and (3.10), we conclude that for $k \ge 1$ and $t \ge t_k$

$$
\begin{aligned}
\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1} \times (1,\infty))} &\le C(t^{\epsilon - \frac{n}{2} - \frac{n}{2}(1-\frac{1}{r}) + \frac{\beta}{2}} + t^{\frac{\beta}{2} - \frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r}) + \frac{\beta}{2}}) \\
&+ C t^{-\frac{n-1}{2}} \sum_{\ell=\ell_0}^{2+k} Q_{r,\ell}(t) g_\ell(t),
\end{aligned} \tag{3.11}
$$

where $n \ge 3$, $\epsilon \in (0, \frac{1}{2})$, and the definitions of $Q_{r,\ell}(t)$, $g_\ell(t)$ ($\ell = \ell_0, \ell_0 + 1, \cdots, 2+k$) are given in the proof of (3.7).

On the other hand, setting $n \ge 3$ and $k \ge 1$, using (1.2) with $n \ge 3$, we get for $t \ge t_k$

$$
\begin{aligned}
\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1} \times (0,1])} &\le \|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1} \times (0,1])} \\
&\le \|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \\
&\le C t^{-\frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})}.
\end{aligned} \tag{3.12}
$$

Combining (3.11) and (3.12), we conclude that for $k \geq 1$ and $t \geq t_k$

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq C(t^{\epsilon - \frac{n}{2} - \frac{n}{2}(1-\frac{1}{r})} + t^{\frac{\beta}{2} - \frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})} + t^{-\frac{2+k}{2} - \frac{n}{2}(1-\frac{1}{r})})$$
$$+ Ct^{-\frac{n-1}{2}} \sum_{\ell=\ell_0}^{2+k} Q_{r,\ell}(t) g_\ell(t) \leq CQ_{r,2+k}(t) + Ct^{-\frac{n-1}{2}} \sum_{\ell=\ell_0}^{2+k} Q_{r,\ell}(t) g_\ell(t),$$

from which we have for $t \geq t_k$

$$\begin{aligned} & Q_{r,2+k}(t^{-1}) \|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \\ & \leq C + Ct^{-\frac{n-1}{2}} Q_{r,2+k}(t^{-1}) \sum_{\ell=\ell_0}^{2+k} Q_{r,\ell}(t) g_\ell(t) \\ & \leq C + Ct^{-\frac{n-1}{2}} Q_{r,2+k}(t^{-1}) Q_{r,\ell_0}(t) \sum_{\ell=\ell_0}^{2+k} g_\ell(t). \end{aligned} \tag{3.13}$$

Observe that $\Theta(n,k,\epsilon) = \begin{cases} n-1 & \text{if } k=1, \\ n-2\epsilon & \text{if } k \geq 2, \end{cases}$ $\ell_0 = \frac{1+k-[\frac{1+k}{2}]}{2}$, and

$$\begin{aligned} & t^{-\frac{n-1}{2}} Q_{r,2+k}(t^{-1}) Q_{r,\ell_0}(t) \\ & = t^{-\frac{n-1}{2} - \min\{\frac{\Theta(n,k,\epsilon)}{2}, \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2}\} + \min\{\frac{\Theta(n,k,\epsilon)}{2}, \frac{2+k}{2} - \frac{\beta}{2}\}} =: t^{-\chi}. \end{aligned} \tag{3.14}$$

A direct calculation shows that for $k \geq 1$ and $0 < \beta < 1$,

$$\chi = \frac{n-1}{2} > 0 \quad \text{if} \quad \frac{\Theta(n,k,\epsilon)}{2} \leq \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2}. \tag{3.15}$$

If $\frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} < \frac{\Theta(n,k,\epsilon)}{2} \leq \frac{2+k}{2} - \frac{\beta}{2}$, then

$$\begin{aligned} \chi &= \begin{cases} \frac{n-1}{2} + \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} - \frac{n-1}{2} & \text{if } k=1, \\ \frac{n-1}{2} + \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} - (\frac{n}{2} - \epsilon) & \text{if } k \geq 2, \end{cases} \\ &= \begin{cases} \frac{1}{2}(1-\beta) & \text{if } k=1, \\ \epsilon + \frac{1}{2}(1-\beta) & \text{if } k=2, \\ \epsilon + \frac{1}{2}(\frac{k-1}{2} - \beta) + \frac{1}{2}\Big(\frac{1+k}{2} - [\frac{1+k}{2}]\Big) & \text{if } k \geq 3, \end{cases} \\ &\geq \frac{1}{2}(1-\beta) > 0. \end{aligned} \tag{3.16}$$

If $\frac{\Theta(n,k,\epsilon)}{2} > \frac{2+k}{2} - \frac{\beta}{2} \Longleftrightarrow \begin{cases} n-2 > 2-\beta & \text{if } k=1, \\ n-2 > 2\epsilon + k - \beta & \text{if } k \geq 2, \end{cases}$ then

$$\begin{aligned}\chi &= \frac{n-1}{2} + \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} - \Big(\frac{2+k}{2} - \frac{\beta}{2}\Big) \\ &= \frac{1}{2}\big(n-2-[\frac{1+k}{2}]\big) \\ &> \begin{cases} \frac{1}{2}(1-\beta) & \text{if } k=1, \\ \frac{1}{2}(2\epsilon + k - \beta - [\frac{1+k}{2}]) & \text{if } k \geq 2, \end{cases} \\ &\geq \frac{1}{2}(1-\beta) > 0. \end{aligned} \tag{3.17}$$

From (3.13)–(3.17), we conclude for $k \geq 1$ and $t \geq t_k$

$$g_{2+k}(t) \leq C + C_0 t^{-\chi} \sum_{\ell=\ell_0}^{1+k} g_\ell(t) + C_0 t^{-\chi} g_{2+k}(t), \tag{3.18}$$

where $\chi > 0$ and $g_\ell(t)$ $(\ell = \ell_0, \ell_0 + 1, \cdots, 2+k)$ is from the proof of (3.7).

Note that there exists $\widetilde{t}_k \geq t_k$ such that $C_0 \widetilde{t}_k^{-\chi} \leq \frac{1}{2}$ in (3.18). Then using (3.18), we get for $n \geq 3$ and $t \geq \widetilde{t}_k$

$$g_{2+k}(t) \leq C_1 + 2C_0 t^{-\chi} \sum_{\ell=1}^{1+k} g_\ell(t). \tag{3.19}$$

Set $S_m(t) = \sum_{\ell=1}^{m} g_\ell(t)$, $m \geq 1$. Observe that $g_{2+k}(t) = S_{2+k}(t) - S_{1+k}(t)$. Then (3.19) yields for $k \geq 1$ and $t \geq \widetilde{t}_k$

$$\begin{aligned} S_{2+k}(t) &\leq C_1 + (1 + 2C_0 t^{-\chi}) S_{1+k}(t) \\ &\leq C_1 + (1 + 2C_0 t^{-\chi})[C_1 + (1 + 2C_0 t^{-\chi}) S_k(t)] \\ &\quad \vdots \\ &\leq C[1 + S_1(t)] \leq C, \end{aligned} \tag{3.20}$$

where $n \geq 3$, $S_1(t) = \sum_{\ell=1}^{1} g_\ell(t) = g_1(t) \leq C$ (see [24]).

Note that $S_{1+k}(t) \leq S_{2+k}(t)$. Using (3.19) and (3.20), we get for $n \geq 3$ and $t \geq \widetilde{t}_k$

$$g_{2+k}(t) \leq C_1 + 2C_0 t^{-\chi} S_{1+k}(t) \leq C_1 + 2C_0 t^{-\chi} S_{2+k}(t) \leq C,$$

which implies

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \le Ct^{-\min\{\frac{\Theta(n,k,\epsilon)}{2}, \frac{2+k}{2}-\frac{\beta}{2}\}-\frac{n}{2}(1-\frac{1}{r})},$$

which is (1.4) by the choice of $\Theta(n,k,\epsilon)$, $\epsilon \in (0, \frac{1}{2})$.

In the next arguments, we always assume $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, $n \ge 2$. Let $1 < r \le \infty$ and $k \ge 1$. Using (1.3) with $n \ge 2$, following the proofs of (3.7), (3.9) and (3.10), we get for $t \ge t_k$

$$\begin{aligned} &\Big\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \partial_{x_n}^{2+k} \mathcal{M}(\cdot, y, t-s) P(u\cdot\nabla u)(y,s)dyds\Big\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))} \\ &\le C\sum_{\ell=0}^{1+k} \int_{\frac{t}{2}}^{t} ((t-s)^{-\frac{1}{2}+\frac{\beta}{2}} + (t-s)^{-\frac{1}{2}}) s^{-1-\frac{\ell}{2}-\frac{1+k-\ell}{2}-n(1-\frac{1}{2r})} ds \\ &\quad + C\int_{\frac{t}{2}}^{t} (t-s)^{-1+\frac{\beta}{2}} s^{-1-\frac{1+k}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} ds + C\int_{\frac{t}{2}}^{t} (t-s)^{-1+\epsilon} s^{-1-n(1-\frac{1}{2r})} ds \\ &\quad + C\sum_{\ell=0}^{1+[\frac{1+k}{2}]} g_{2+k-\ell}(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{n+1}{2}} \overline{Q}_{r,2+k-\ell}(s) ds \\ &\le Ct^{\epsilon-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + Ct^{-\frac{n}{2}} \sum_{\ell=\ell_0}^{2+k} \overline{Q}_{r,\ell}(t) \bar{g}_\ell(t); \end{aligned} \tag{3.7$'$}$$

$$\begin{aligned} &\Big\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \nabla_{x'}^{2+k} \mathcal{M}(\cdot, y, t-s) P(u\cdot\nabla u)(y,s)dyds\Big\|_{L^r(\mathbb{R}^n_+)} \\ &\le Ct^{\epsilon-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + Ct^{-\frac{n}{2}} \sum_{\ell=\ell_0}^{2+k} \overline{Q}_{r,\ell}(t) \bar{g}_\ell(t); \end{aligned} \tag{3.9$'$}$$

and

$$\begin{aligned} &\Big\|x_n^\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \nabla_{x}^{2+k} \mathcal{M}(\cdot, y, t-s) Pu(y,s)\cdot\nabla u(y,s)dyds\Big\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))} \\ &\le Ct^{\epsilon-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})} + Ct^{-\frac{n}{2}} \sum_{\ell=\ell_0}^{2+k} \overline{Q}_{r,\ell}(t) \bar{g}_\ell(t), \end{aligned} \tag{3.10$'$}$$

where $\epsilon \in (0, \frac{1}{2})$, $\ell_0 = 1 + k - [\frac{1+k}{2}]$ is from the proof of (3.7),

$$\bar{g}_\ell(t) = \sup_{0<s\le t} [\overline{Q}_{r,\ell}(s^{-1}) \|y_n^\beta \nabla^\ell u(s)\|_{L^r(\mathbb{R}^n_+)}],$$

$$\overline{Q}_{r,\ell}(t) = t^{-\min\{\frac{n}{2}-\epsilon, \frac{\ell}{2}-\frac{\beta}{2}\}-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}.$$

From (3.1), (3.7′), (3.9′) and (3.10′), we infer for $k \geq 1$ and $t \geq t_k$

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1}\times(1,\infty))} \leq Ct^{\epsilon-1-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n}{2}} \sum_{\ell=\ell_0}^{2+k} \overline{Q}_{r,\ell}(t)\bar{g}_\ell(t). \tag{3.11′}$$

Using (1.3) with $n \geq 2$ yields for $k \geq 1$ and $t \geq t_k$

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1}\times(0,1])} \leq \|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^{n-1}\times(0,1])} \leq \|\nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}-\frac{2+k}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{3.12′}$$

Combining (3.11′) and (3.12′), we infer for $k \geq 1$ and $t \geq t_k$

$$\begin{aligned} &\overline{Q}_{r,2+k}(t^{-1})\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \\ &\leq C + Ct^{-\frac{n}{2}} \overline{Q}_{r,2+k}(t^{-1}) \sum_{\ell=\ell_0}^{2+k} \overline{Q}_{r,\ell}(t)\bar{g}_\ell(t)) \\ &\leq C + Ct^{-\frac{n}{2}} \overline{Q}_{r,2+k}(t^{-1})\overline{Q}_{r,\ell_0}(t) \sum_{\ell=\ell_0}^{2+k} \bar{g}_\ell(t). \end{aligned} \tag{3.13′}$$

Observe that $\ell_0 = \frac{1+k-[\frac{1+k}{2}]}{2}$, and for $\epsilon \in (0, \frac{1}{2})$,

$$\begin{aligned} &t^{-\frac{n}{2}} \overline{Q}_{r,2+k}(t^{-1})\overline{Q}_{r,\ell_0}(t) \\ &= t^{-\frac{n}{2}-\min\{\frac{n}{2}-\epsilon, \frac{1+k-[\frac{1+k}{2}]}{2}-\frac{\beta}{2}\}+\min\{\frac{n}{2}-\epsilon, \frac{2+k}{2}-\frac{\beta}{2}\}} =: t^{-\bar{\chi}}. \end{aligned} \tag{3.14′}$$

A direct computation shows that for $n \geq 2$ and $k \geq 1$

$$\bar{\chi} = \frac{n}{2} > 0 \quad \text{if} \quad \frac{n}{2} - \epsilon \leq \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2}; \tag{3.15′}$$

$$\bar{\chi} = \frac{n}{2} + \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} - (\frac{n}{2} - \epsilon) > \frac{1}{2}\big(\frac{1+k}{2} - \beta\big) > 0 \tag{3.16′}$$

if $\frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} < \frac{n}{2} - \epsilon \leq \frac{2+k}{2} - \frac{\beta}{2}$;

$$\begin{aligned}\bar{\chi} &= \frac{n}{2} + \frac{1+k-[\frac{1+k}{2}]}{2} - \frac{\beta}{2} - (\frac{2+k}{2} - \frac{\beta}{2}) \\ &= \frac{1}{2}\big(n-1-[\frac{1+k}{2}]\big) \\ &> \frac{1}{2}(2\epsilon+1+k-\beta-[\frac{1+k}{2}]) \\ &> \frac{1}{2}\big(\frac{1+k}{2}-\beta\big) > 0 \end{aligned} \tag{3.17′}$$

if $\frac{n}{2} - \epsilon > \frac{2+k}{2} - \frac{\beta}{2} \Longleftrightarrow n-1 > 2\epsilon+1+k-\beta$.

From (3.13′)–(3.17′), we conclude for $k \geq 1$ and $t \geq t_k$

$$\bar{g}_{2+k}(t) \leq C + \overline{C}_0 t^{-\bar{\chi}} \sum_{\ell=\ell_0}^{1+k} \bar{g}_\ell(t) + \overline{C}_0 t^{-\bar{\chi}} \bar{g}_{2+k}(t), \tag{3.18′}$$

where $\bar{g}_\ell(t)$ $(\ell = \ell_0, \ell_0+1, \cdots, 2+k)$ is from the proof of (3.7′).

Note that there exists $\bar{t}_k \geq t_k$ such that $\overline{C}_0 \bar{t}_k^{-\bar{\chi}} \leq \frac{1}{2}$ in (3.18′). Then using (3.18′), we get for $n \geq 2$ and $t \geq \bar{t}_k$

$$\bar{g}_{2+k}(t) \leq \overline{C}_1 + 2\overline{C}_0 t^{-\bar{\chi}} \sum_{\ell=1}^{1+k} \bar{g}_\ell(t). \tag{3.19′}$$

Set $\overline{S}_m(t) = \sum_{\ell=1}^{m} \bar{g}_\ell(t)$, $m \geq 1$. Observe that $\bar{g}_{2+k}(t) = \overline{S}_{2+k}(t) - \overline{S}_{1+k}(t)$. Then (3.19′) yields for $k \geq 1$ and $t \geq \bar{t}_k$

$$\begin{aligned}\overline{S}_{2+k}(t) &\leq \overline{C}_1 + (1+2\overline{C}_0 t^{-\bar{\chi}})\overline{S}_{1+k}(t) \\ &\leq \overline{C}_1 + (1+2\overline{C}_0 t^{-\bar{\chi}})[\overline{C}_1 + (1+2\overline{C}_0 t^{-\chi})\overline{S}_k(t)] \\ &\quad\vdots \\ &\leq \overline{C}[1+\overline{S}_1(t)] \leq \overline{C}, \end{aligned} \tag{3.20′}$$

where $\overline{S}_1(t) = \bar{g}_1(t) \leq C$ (see [24]).

Note that $\overline{S}_{1+k}(t) \leq \overline{S}_{2+k}(t)$. Using (3.19′) and (3.20′), we get for $n \geq 2$ and $t \geq \bar{t}_k$

$$\bar{g}_{2+k}(t) \leq \overline{C}_1 + 2\overline{C}_0 t^{-\bar{\chi}} \overline{S}_{1+k}(t) \leq \overline{C}_1 + 2\overline{C}_0 t^{-\bar{\chi}} \overline{S}_{2+k}(t) \leq \overline{C}_\epsilon,$$

which implies

$$\|x_n^\beta \nabla^{2+k} u(t)\|_{L^r(\mathbb{R}^n_+)} \leq \overline{C}_\epsilon t^{-\min\{\frac{n}{2}-\epsilon, \frac{2+k}{2}-\frac{\beta}{2}\} - \frac{1}{2} - \frac{n}{2}(1-\frac{1}{r})}$$

holds under the assumption $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$ for $n \geq 2$. This shows (1.5) is valid. Here $\overline{C}_\epsilon = \overline{C}(n, k, \epsilon, r, \beta, u_0)$. □

4. L^1-decay estimates for the Stokes flow

Firstly recall the generalized Stokes formula. For details, see [42]. Let D be a bounded domain in $\mathbb{R}^n$ with the smooth boundary ∂D. We denote by ν the unit outer-normal to ∂D. For $1 < r < 1$, we define the space $E_r(D)$ by

$$E_r(D) = \{v \in L^r(D)|\ div\, v \in L^r(D)\}.$$

Equipped with the norm $\|v\|_{E_r(D)} = \|v\|_{L^r(D)} + \|div\ v\|_{L^r(D)}$, $E_r(D)$ is a Banach space. For $v \in E_r(D)$, $v \cdot \nu$ is well-defined as an element of $[W^{1-\frac{1}{r'},r'}(\partial D)]^*$, the dual space of $W^{1-\frac{1}{r'},r'}(\partial D)$, where $\frac{1}{r'} = 1 - \frac{1}{r}$. There holds the generalized Stokes formula for all $f \in E_r(D)$, $g \in W^{1,r'}(D)$,

$$\int\limits_D f \cdot \nabla g dx + \int\limits_D g\ div\, f dx = \langle f \cdot \nu, g\rangle|_{\partial D}, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle|_{\partial D}$ denotes the duality between $[W^{1-\frac{1}{r'},r'}(\partial D)]^*$ and $W^{1-\frac{1}{r'},r'}(\partial D)$. Moreover, it holds for all $v \in E_r(D)$

$$\|v \cdot \nu\|_{[W^{1-\frac{1}{r'},r'}(\partial D)]^*} \le C(D, r)\|v\|_{E_r(D)}.$$

It is proved in [32] by using the Fourier transforms that $\int_{\mathbb{R}^n} f(x)dx = 0$ for all $f \in L^1(\mathbb{R}^n) \cap L^r_\sigma(\mathbb{R}^n)$ with some $1 < r < \infty$. It is reasonable to ask whether the same result holds in the half-space $\mathbb{R}^n_+$, the following lemma gives an affirmative answer to this question.

Lemma 4.1. *Let $a \in L^1(\mathbb{R}^n_+) \cap L^r_\sigma(\mathbb{R}^n_+)$ with some $1 < r < \infty$. Then*

$$\int\limits_{\mathbb{R}^n_+} a(x)dx = 0.$$

Proof. Take $\chi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \chi \le 1$ with $\chi(x) = 1$ for $|x| \le 1$; $\chi(x) = 0$ for $|x| \ge 2$. Set $\chi_R(x) = \chi(\frac{x}{R})$, $R > 0$. It is not difficult to find that $x_\ell\chi_R(x) \in W^{1,r'}(\mathbb{R}^n_+ \cap B_{3R}(0))$ with $\ell = 1, 2, \cdots, n$, $\frac{1}{r'} = 1 - \frac{1}{r}$, $1 < r < \infty$. Note that $a \in L^1(\mathbb{R}^n_+) \cap L^r_\sigma(\mathbb{R}^n_+)$. Using (4.1), we get for $\ell = 1, 2, \cdots, n$,

$$\int\limits_{\mathbb{R}^n_+} a(x) \cdot \nabla(x_\ell\chi_R(x))dx$$

$$= \int\limits_{\mathbb{R}^n_+ \cap B_{3R}(0)} a(x) \cdot \nabla(x_\ell\chi_R(x))dx$$

$$
\begin{aligned}
&= - \int\limits_{\mathbb{R}^n_+ \cap B_{3R}(0)} x_\ell \chi_R(x) \; div \; a(x) \; dx + \langle a(x) \cdot \nu, \; x_\ell \chi_R(x) \rangle |_{\partial(\mathbb{R}^n_+ \cap B_{3R}(0))} \\
&= \langle a \cdot \nu, x_\ell \chi_R(x) \rangle |_{\partial \mathbb{R}^n_+ \cap B_{3R}(0)} + \langle a \cdot \nu, x_\ell \chi_R(x) \rangle |_{\mathbb{R}^n_+ \cap \partial B_{3R}(0)} \\
&= 0. \qquad (4.2)
\end{aligned}
$$

Now we estimate the term $\int_{\mathbb{R}^n_+} a \cdot \nabla(x_\ell \chi_R(x)) dx$ in (4.2). After a direct calculation, we find for $\ell = 1, 2, \cdots, n$

$$
\int\limits_{\mathbb{R}^n_+} a(x) \cdot \nabla(x_\ell \chi_R(x)) dx = \int\limits_{\mathbb{R}^n_+} (a_\ell(x) \chi_R(x) + \frac{x_\ell}{R} a(x) \cdot \nabla \chi(\frac{x}{R})) dx. \qquad (4.3)
$$

Note that $a \in L^1(\mathbb{R}^n)$. We find for $\ell = 1, 2, \cdots, n$

$$
\begin{aligned}
&| \int\limits_{\mathbb{R}^n_+} a_\ell(x) \chi_R(x) dx - \int\limits_{\mathbb{R}^n_+} a_\ell(x) dx | \\
&\leq \int\limits_{\mathbb{R}^n_+} |a_\ell(x)| |\chi_R(x) - 1| dx \\
&\leq \int\limits_{\mathbb{R}^n_+ \cap \{|x| > R\}} |a_\ell(x)| dx \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty; \qquad (4.4)
\end{aligned}
$$

and by the choice of the cut-off function χ_R,

$$
\begin{aligned}
&| \int\limits_{\mathbb{R}^n_+} \frac{x_\ell}{R} a(x) \cdot \nabla \chi(\frac{x}{R}) dx | \\
&\leq 2 \sup_{R < |y| \leq 2R} |\nabla \chi(y)| \int\limits_{\mathbb{R}^n_+ \cap \{R < |y| \leq 2R\}} |a(x)| dx \\
&\leq 2 \sup_{y \in \mathbb{R}^n} |\nabla \chi(y)| \int\limits_{\mathbb{R}^n_+ \cap \{|y| > R\}} |a(x)| dx \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty. \qquad (4.5)
\end{aligned}
$$

Inserting (4.4) and (4.5) into (4.3), we conclude for $\ell = 1, 2, \cdots, n$

$$
\lim_{R \longrightarrow \infty} \int\limits_{\mathbb{R}^n_+} a(x) \cdot \nabla(x_\ell \chi_R(x)) dx = \int\limits_{\mathbb{R}^n_+} a_\ell(x) dx. \qquad (4.6)
$$

Combining (4.2) and (4.6), we deduce $\int_{\mathbb{R}^n_+} a_\ell(x) dx = 0$, for $\ell = 1, 2, \cdots, n$. $\square$

Thanks to Lemma 4.1, now we focus and work on Theorem 1.3.

Proof of Theorem 1.3. Let a be given in Theorem 1.3. Then it holds for $x \in \mathbb{R}^n_+$ and $t > 0$ (see [40,41])

$$\partial^k [e^{-tA}a](x) = \int\limits_{\mathbb{R}^n_+} \partial_x^k \mathcal{M}(x,y,t)a(y)dy, \quad k = 0,1,2,\cdots, \tag{4.7}$$

where the definition of $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$ is given in the proof of Theorem 1.1 in Section 2. Combining Lemma 4.1 and (4.7) yields for $0 \le \alpha \le 1$ and $t > 0$

$$\begin{aligned}
|[e^{-tA}a](x)| &= |\int\limits_{\mathbb{R}^n_+} [\mathcal{M}(x,y,t) - \mathcal{M}(x,0,t)]a(y)dy| \\
&\le \int\limits_{\mathbb{R}^n_+} |\mathcal{M}(x,y,t) - \mathcal{M}(x,0,t)|^{1-\alpha} |\mathcal{M}(x,y,t) - \mathcal{M}(x,0,t)|^{\alpha} |a(y)|dy \\
&\le \int\limits_{\mathbb{R}^n_+} (|\mathcal{M}(x,y,t)| + |\mathcal{M}(x,0,t)|)^{1-\alpha} |\int_0^1 \partial_y \mathcal{M}(x,sy,t)ds|^{\alpha} |y|^{\alpha} |a(y)|dy \\
&\le C \int\limits_{\mathbb{R}^n_+} (G_t(x-y) + G_t(x) + |\mathcal{M}^*(x,y,t)| + |\mathcal{M}^*(x,0,t)|)^{1-\alpha} \\
&\quad \times (|\partial_y G_t(x - s_0 y)| + |\partial_y \mathcal{M}^*(x, s_0 y, t)|)^{\alpha} |y|^{\alpha} |a(y)|dy, \quad s_0 \in (0,1).
\end{aligned} \tag{4.8}$$

Using the estimates on $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$, which are given in the proof of Theorem 1.1 in Section 2, and from (4.8), we have for $0 \le \alpha \le 1$ and $t > 0$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n-1}} |[e^{-tA}a](x)|dx' \\
&\le \int\limits_{\mathbb{R}^n_+} \Big(\int\limits_{\mathbb{R}^{n-1}} (G_t(x-y) + G_t(x) + |\mathcal{M}^*(x,y,t)| + |\mathcal{M}^*(x,0,t)|)dx' \Big)^{1-\alpha} \\
&\quad \times \Big(\int\limits_{\mathbb{R}^{n-1}} (|\partial_y G_t(x - s_0 y)| + |\partial_y \mathcal{M}^*(x, s_0 y, t)|)dx' \Big)^{\alpha} |y|^{\alpha} |a(y)|dy \\
&\le C \int\limits_{\mathbb{R}^n_+} \Big(\|G_t(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})} + \|G_t(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})} \\
&\quad + \int\limits_{\mathbb{R}^{n-1}} (|x'| + x_n + \sqrt{t}\,)^{-n} dx' \Big)^{1-\alpha} \Big(\int\limits_{\mathbb{R}^{n-1}} |\partial_y G_t(x - s_0 y)|dx' \\
&\quad + t^{-\frac{1_n}{2}} \int\limits_{\mathbb{R}^{n-1}} (|x'| + x_n + s_0 y_n + \sqrt{t}\,)^{-n-|1'|} dx' \Big)^{\alpha} |y|^{\alpha} |a(y)|dy
\end{aligned}$$

$$\leq C \int\limits_{\mathbb{R}^n_+} \Big(\|G_t(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})} + \|G_t(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})} + (x_n + \sqrt{t}\,)^{-1} \Big)^{1-\alpha}$$

$$\times \Big(\int\limits_{\mathbb{R}^{n-1}} |\partial_y G_t(x', x_n - s_0 y_n)| dx' + t^{-\frac{1_n}{2}} (x_n + \sqrt{t}\,)^{-|1'|-1} \Big)^{\alpha} |y|^{\alpha} |a(y)| dy,$$

from which we get for $0 \leq \alpha \leq 1$ and $t > 0$

$$\int\limits_0^{\sqrt{t}} \int\limits_{\mathbb{R}^{n-1}} |[e^{-tA} a](x)| dx' dx_n$$

$$\leq C \int\limits_{\mathbb{R}^n_+} \Big(\int\limits_0^{\sqrt{t}} [\|G_t(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})}$$

$$+ \|G_t(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})} + (x_n + \sqrt{t}\,)^{-1}] dx_n \Big)^{1-\alpha}$$

$$\times \Big(\int\limits_0^{\sqrt{t}} \int\limits_{\mathbb{R}^{n-1}} |\partial_y G_t(x', x_n - s_0 y_n)| dx' dx_n$$

$$+ t^{-\frac{1_n}{2}} \int\limits_0^{\sqrt{t}} (x_n + \sqrt{t}\,)^{-|1'|-1} dx_n \Big)^{\alpha} |y|^{\alpha} |a(y)| dy$$

$$\leq C \Big(\int\limits_{-\infty}^{\infty} [\|G_t(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})}$$

$$+ \|G_t(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})}] dx_n + \int\limits_0^{\sqrt{t}} (x_n + \sqrt{t}\,)^{-1} dx_n \Big)^{1-\alpha}$$

$$\times \Big(\int\limits_{-\infty}^{\infty} \int\limits_{\mathbb{R}^{n-1}} |\partial G_t(x)| dx + t^{-\frac{1_n}{2}} \int\limits_0^{\sqrt{t}} (x_n + \sqrt{t}\,)^{-|1'|-1} dx_n \Big)^{\alpha} \int\limits_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy$$

$$\leq C(1 + \log_e(2\sqrt{t}\,) - \log_e \sqrt{t}\,)^{1-\alpha} (t^{-\frac{1}{2}} + L(t))^{\alpha} \int\limits_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy$$

$$\leq C t^{-\frac{\alpha}{2}} \int\limits_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy, \tag{4.9}$$

where

$$L(t) = \begin{cases} t^{-\frac{1}{2}}(\log_e(2\sqrt{t}\,) - \log_e \sqrt{t}\,) & \text{if } |1'| = 0, \\ t^{-\frac{1}{2}} & \text{if } |1'| = 1, \end{cases}$$
$$= \begin{cases} t^{-\frac{1}{2}} \log_e 2 & \text{if } |1'| = 0, \\ t^{-\frac{1}{2}} & \text{if } |1'| = 1. \end{cases}$$

In the next arguments, we assume the integer $k = |(k', k_n)|$ and $k \geq 1$.

Using (4.7), we have for $t > 0$

$$\begin{aligned}
&|\partial^k[e^{-tA}a](x)| \\
&= |\int_{\mathbb{R}^n_+} \partial^k_x[\mathcal{M}(x,y,t) - \mathcal{M}(x,0,t)]a(y)dy| \\
&\leq \int_{\mathbb{R}^n_+} |\partial^k_x\mathcal{M}(x,y,t) - \partial^k_x\mathcal{M}(x,0,t)|^{1-\alpha}|\partial^k_x\mathcal{M}(x,y,t) - \partial^k_x\mathcal{M}(x,0,t)|^{\alpha}|a(y)|dy \\
&\leq \int_{\mathbb{R}^n_+} (|\partial^k_x\mathcal{M}(x,y,t)| + |\partial^k_x\mathcal{M}(x,0,t)|)^{1-\alpha}|\partial^k_x \int_0^1 \partial_y\mathcal{M}(x,sy,t)ds|^{\alpha}|y|^{\alpha}|a(y)|dy \\
&\leq C\int_{\mathbb{R}^n_+} (|\partial^k_x G_t(x-y)| + \partial^k_x G_t(x)| + |\partial^k_x\mathcal{M}^*(x,y,t)| + |\partial^k_x\mathcal{M}^*(x,0,t)|)^{1-\alpha} \\
&\quad \times (|\partial^k_x\partial_y G_t(x-s_1y)| + |\partial^k_x\partial_y\mathcal{M}^*(x,s_1y,t)|)^{\alpha}|y|^{\alpha}|a(y)|dy, \quad s_1 \in (0,1).
\end{aligned} \tag{4.10}$$

Using the estimates on $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$, which are given in the proof of Theorem 1.1 in Section 2, and from (4.10), we have for $0 \leq \alpha \leq 1$ and $t > 0$

$$\begin{aligned}
&\|\partial^k[e^{-tA}a]\|_{L^1(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} |\partial^k[e^{-tA}a](x)|dx \\
&\leq \int_{\mathbb{R}^n_+}\Big(\int_{\mathbb{R}^n_+} (|\partial^k_x G_t(x-y)| + |\partial^k_x G_t(x)| \\
&\quad + |\partial^k_x\mathcal{M}^*(x,y,t)| + |\partial^k_x\mathcal{M}^*(x,0,t)|)dx\Big)^{1-\alpha} \\
&\quad \times \Big(\int_{\mathbb{R}^n_+} (|\partial^k_x\partial_y G_t(x-s_1y)| + |\partial^k_x\partial_y\mathcal{M}^*(x,s_1y,t)|)dx\Big)^{\alpha}|y|^{\alpha}|a(y)|dy \\
&\leq C\int_{\mathbb{R}^n_+}\Big(\|\partial^k_x G_t\|_{L^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n_+} (x_n + \sqrt{t}\,)^{-k_n} \\
&\quad \times (|x'| + x_n + \sqrt{t}\,)^{-n-|k'|}dx'\Big)^{1-\alpha}\Big(\int_{\mathbb{R}^n_+} |\partial^k_x\partial_y G_t(x-s_1y)|dx
\end{aligned}$$

$$+ t^{-\frac{1_n}{2}} \int_{\mathbb{R}^{n-1}} (x_n + \sqrt{t})^{-k_n}$$
$$\times (|x'| + x_n + s_1 y_n + \sqrt{t})^{-n-|1'|-|k'|} dx' dx_n \Big)^{\alpha} |y|^{\alpha} |a(y)| dy$$
$$\leq C\Big(t^{-\frac{k}{2}} + \int_0^{\infty} (x_n + \sqrt{t})^{-1-k} dx_n\Big)^{1-\alpha}$$
$$\times \Big(\|\partial^{k+1} G_t\|_{L^1(\mathbb{R}^n)} + t^{-\frac{1_n}{2}} \int_0^{\infty} (x_n + \sqrt{t})^{-|1'|-1-k} dx_n\Big)^{\alpha} \int_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy$$
$$\leq C t^{-\frac{k(1-\alpha)}{2} - \frac{(k+1)\alpha}{2}} \int_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy$$
$$\leq C t^{-\frac{k}{2} - \frac{\alpha}{2}} \int_{\mathbb{R}^n_+} |y|^{\alpha} |a(y)| dy. \tag{4.11}$$

From (4.9) and (4.11), we complete the proof of Theorem 1.3. □

5. L^1-behavior for Navier–Stokes flows

Lemma 5.1. *Let $1 < r < \infty$, and suppose $b(x,t) \in L^1(\mathbb{R}^n_+) \cap L^r_\sigma(\mathbb{R}^n_+)$ with almost all $t > 0$. Then for $0 \leq \alpha \leq 1$ and $t > 0$*

$$\Big\| \int_0^{\frac{t}{2}} e^{-(t-s)A} b(\cdot, s) ds \Big\|_{L^1(\mathbb{R}^{n-1} \times (0, \sqrt{t}))} \leq C t^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \| |y|^{\alpha} b(\cdot, s)\|_{L^1(\mathbb{R}^n_+)} ds; \tag{5.1}$$

$$\Big\| \int_{\frac{t}{2}}^{t} e^{-(t-s)A} b(\cdot, s) ds \Big\|_{L^1(\mathbb{R}^{n-1} \times (0, \sqrt{t}))} \leq C t^{1-\frac{\alpha}{2}} \sup_{\frac{t}{2} \leq s \leq t} \| |y|^{\alpha} b(\cdot, s)\|_{L^1(\mathbb{R}^n_+)}. \tag{5.2}$$

Proof. Using Lemma 4.1, together with the assumption on b, we have for almost all $t > 0$

$$\int_{\mathbb{R}^n_+} b(x,t) dx = 0. \tag{5.3}$$

From (4.7), we find for $x \in \mathbb{R}^n_+$ and $t > 0$

$$\Big[\int_{\frac{t}{2}}^{t} e^{-(t-s)A} b(\cdot, s) ds\Big](x) = \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \mathcal{M}(x, y, t-s) b(y, s) dy ds, \tag{5.4}$$

and

$$\Big[\int_0^{\frac{t}{2}} e^{-(t-s)A} b(\cdot,s)ds\Big](x) = \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s)b(y,s)dyds. \tag{5.5}$$

Combining (5.3) and (5.4) yields for $0 \le \alpha \le 1$ and $t > 0$

$$\begin{aligned}
&\Big|\Big[\int_{\frac{t}{2}}^{t} e^{-(t-s)A} b(\cdot,s)ds\Big](x)\Big| \\
&= \Big|\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} [\mathcal{M}(x,y,t-s) - \mathcal{M}(x,0,t-s)]b(y,s)dyds\Big| \\
&\le \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} |\mathcal{M}(x,y,t-s) - \mathcal{M}(x,0,t-s)|^{1-\alpha} \\
&\quad \times |\mathcal{M}(x,y,t-s) - \mathcal{M}(x,0,t-s)|^{\alpha}|b(y,s)|dyds \\
&\le \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} (|\mathcal{M}(x,y,t-s)| + |\mathcal{M}(x,0,t-s)|)^{1-\alpha} \\
&\quad \times \Big|\int_0^1 \nabla_y \mathcal{M}(x,\tau y,t-s)d\tau\Big|^{\alpha} |y|^{\alpha}|b(y,s)|dyds \\
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} (G_{t-s}(x-y) + G_{t-s}(x) + |\mathcal{M}^*(x,y,t-s)| + |\mathcal{M}^*(x,0,t-s)|)^{1-\alpha} \\
&\quad \times (|\partial_y G_{t-s}(x-\tau_0 y)| + |\partial_y \mathcal{M}^*(x,\tau_0 y,t-s)|)^{\alpha}|y|^{\alpha}|b(y,s)|dyds,\ \tau_0 \in (0,1).
\end{aligned} \tag{5.6}$$

Using the estimates on $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$, which are given in the proof of Theorem 1.1 in Section 2, together with (5.6), we conclude for $0 \le \alpha \le 1$ and $t > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \Big|\Big[\int_{\frac{t}{2}}^{t} e^{-(t-s)A} b(\cdot,s)ds\Big](x',x_n)\Big|dx' \\
&\le \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\Big(\int_{\mathbb{R}^{n-1}} (G_{t-s}(x-y) + G_{t-s}(x) \\
&\quad + |\mathcal{M}^*(x,y,t-s)| + |\mathcal{M}^*(x,0,t-s)|)dx'\Big)^{1-\alpha}
\end{aligned}$$

$$\times \Big(\int\limits_{\mathbb{R}^{n-1}} (|\partial_y G_{t-s}(x-\tau_0 y)| + |\partial_y \mathcal{M}^*(x, \tau_0 y, t-s)|) dx' \Big)^{\alpha} |y|^{\alpha} |b(y,s)| dy ds$$

$$\leq C \int\limits_{\frac{t}{2}}^{t} \int\limits_{\mathbb{R}^n_+} \Big((\|G_{t-s}(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})} + \|G_{t-s}(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})}$$

$$+ \int\limits_{\mathbb{R}^{n-1}} (|x'| + x_n + \sqrt{t-s}\,)^{-n} dx' \Big)^{1-\alpha} \Big(\int\limits_{\mathbb{R}^{n-1}} |\partial_y G_{t-s}(x - \tau_0 y)| dx'$$

$$+ (t-s)^{-\frac{1_n}{2}} \int\limits_{\mathbb{R}^{n-1}} (|x'| + x_n + \tau_0 y_n + \sqrt{t-s}\,)^{-n-|1'|} dx' \Big)^{\alpha} |y|^{\alpha} |b(y,s)| dy ds$$

$$\leq C \int\limits_{\frac{t}{2}}^{t} \int\limits_{\mathbb{R}^n_+} \Big(\|G_{t-s}(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})} + \|G_{t-s}(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})}$$

$$+ (x_n + \sqrt{t-s}\,)^{-1} \Big)^{1-\alpha} \Big(\int\limits_{\mathbb{R}^{n-1}} |\partial G_{t-s}(x', x_n - \tau_0 y_n)| dx'$$

$$+ (t-s)^{-\frac{1_n}{2}} (x_n + \sqrt{t-s}\,)^{-|1'|-1} \Big)^{\alpha} |y|^{\alpha} |b(y,s)| dy ds,$$

from which we find for $0 \leq \alpha \leq 1$ and $t > 0$

$$\int\limits_{0}^{\sqrt{t}} \int\limits_{\mathbb{R}^{n-1}} \Big| \Big[\int\limits_{\frac{t}{2}}^{t} e^{-(t-s)A} b(\cdot, s) ds \Big] (x', x_n) \Big| dx' dx_n$$

$$\leq C \int\limits_{\frac{t}{2}}^{t} \int\limits_{\mathbb{R}^n_+} \Big(\int\limits_{0}^{\sqrt{t}} [\|G_{t-s}(\cdot, x_n - y_n)\|_{L^1(\mathbb{R}^{n-1})} + \|G_{t-s}(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})}$$

$$+ (x_n + \sqrt{t-s}\,)^{-1}] dx_n \Big)^{1-\alpha} \Big(\int\limits_{0}^{\sqrt{t}} \int\limits_{\mathbb{R}^{n-1}} |\partial G_{t-s}(x', x_n - \tau_0 y_n)| dx' dx_n$$

$$+ (t-s)^{-\frac{1_n}{2}} \int\limits_{0}^{\sqrt{t}} (x_n + \sqrt{t-s}\,)^{-|1'|-1} dx_n \Big)^{\alpha} |y|^{\alpha} |b(y,s)| dy ds$$

$$\leq C \int\limits_{\frac{t}{2}}^{t} \Big(\int\limits_{-\infty}^{\infty} \|G_{t-s}(\cdot, x_n)\|_{L^1(\mathbb{R}^{n-1})} dx_n + \int\limits_{0}^{\sqrt{t}} (x_n + \sqrt{t-s}\,)^{-1} dx_n \Big)^{1-\alpha}$$

$$\times \Big(\int\limits_{-\infty}^{\infty} \int\limits_{\mathbb{R}^{n-1}} |\partial G_{t-s}(x)| dx$$

$$
\begin{aligned}
&+ (t-s)^{-\frac{1_n}{2}} \Big(\int_0^{\sqrt{t}} (x_n + \sqrt{t-s}\,)^{-|1'|-1} dx_n \Big)^{\alpha} \int_{\mathbb{R}^n_+} |y|^{\alpha} |b(y,s)| dy ds \\
&\le C \int_{\frac{t}{2}}^{t} (1 + \log_e(\sqrt{t} + \sqrt{t-s}\,) - \log_e \sqrt{t-s}\,)^{1-\alpha} \\
&\quad \times \big((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} K(\frac{s}{t}) \big)^{\alpha} \int_{\mathbb{R}^n_+} |y|^{\alpha} |b(y,s)| dy ds \\
&\le C g(t) \int_{\frac{t}{2}}^{t} \Big(1 + \log_e \Big(1 + \sqrt{\frac{t}{t-s}} \Big) \Big)^{1-\alpha} ((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} K(\frac{s}{t}))^{\alpha} ds \\
&\le C g(t) t^{1-\frac{\alpha}{2}} \int_{\frac{1}{2}}^{1} (1 + \log_e(1 + \sqrt{\frac{1}{1-\lambda}}\,))^{1-\alpha} ((1-\lambda)^{-\frac{1}{2}} + (1-\lambda)^{-\frac{1}{2}} K(\lambda))^{\alpha} d\lambda,
\end{aligned}
\tag{5.7}
$$

where $g(t) = \sup\limits_{\frac{t}{2} \le s \le t} \int_{\mathbb{R}^n_+} |y|^{\alpha} |b(y,s)| dy$, and $K(\lambda) = \begin{cases} \log_e(1 + \sqrt{\frac{1}{1-\lambda}}\,) & \text{if } |1'| = 0, \\ 1 & \text{if } |1'| = 1. \end{cases}$

Note that $\sqrt{\frac{1}{1-\lambda}} \ge \sqrt{2}$ for all $\frac{1}{2} \le \lambda \le 1$. Whence

$$
\begin{aligned}
\log_e(1 + \sqrt{2}\,) &\le \log_e \Big(1 + \sqrt{\frac{1}{1-\lambda}} \Big) \\
&\le \log_e \Big((1 + \frac{1}{\sqrt{2}}) \sqrt{\frac{1}{1-\lambda}} \Big) \\
&\le \log_e 2 + \log_e \Big(\sqrt{\frac{1}{1-\lambda}} \Big) \\
&\le \log_e \Big(\frac{1}{1-\lambda} \Big) + \frac{1}{2} \log_e \Big(\frac{1}{1-\lambda} \Big) \\
&= -\frac{3}{2} \log_e(1-\lambda)
\end{aligned}
\tag{5.8}
$$

and

$$
\min\{1, \log_e(1+\sqrt{2}\,)\} \le K(\lambda) \le 2 \log_e \Big(1 + \sqrt{\frac{1}{1-\lambda}} \Big) \le -3 \log_e(1-\lambda). \tag{5.9}
$$

A direct calculation shows that

$$\int_{\frac{1}{2}}^{1}(1-\lambda)^{-\frac{\alpha}{2}}[-\log_e(1-\lambda)]d\lambda = \int_{\log_e 2}^{\infty} se^{-s(1-\frac{\alpha}{2})}ds$$
$$= (\frac{2}{2-\alpha})^2 \int_{(1-\frac{\alpha}{2})\log_e 2}^{\infty} \lambda e^{-\lambda}d\lambda$$
$$= (\frac{2}{2-\alpha})^2\big((1+\frac{\alpha}{2})\log_e 2 - 1\big)e^{-(1-\frac{\alpha}{2})\log_e 2}. \tag{5.10}$$

From (5.8)–(5.10), we find

$$\int_{\frac{1}{2}}^{1}(1+\log_e(1+\sqrt{\frac{1}{1-\lambda}}))^{1-\alpha}((1-\lambda)^{-\frac{1}{2}}+(1-\lambda)^{-\frac{1}{2}}K(\lambda))^{\alpha}d\lambda$$
$$\leq C\int_{\frac{1}{2}}^{1}(1-\lambda)^{-\frac{\alpha}{2}}[-\log_e(1-\lambda)]^{\alpha}d\lambda$$
$$\leq C\int_{\frac{1}{2}}^{1}(1-\lambda)^{-\frac{\alpha}{2}}[1-\log_e(1-\lambda)]d\lambda = C(\alpha). \tag{5.11}$$

Inserting (5.11) into (5.7), we obtain for $t > 0$

$$\Big\|\int_{\frac{t}{2}}^{t} e^{-(t-s)A}b(\cdot,s)ds\Big\|_{L^1(\mathbb{R}^{n-1}\times(0,\sqrt{t}\,))}$$
$$\leq \int_0^{\sqrt{t}}\int_{\mathbb{R}^{n-1}}\Big|\Big[\int_{\frac{t}{2}}^{t} e^{-(t-s)A}b(\cdot,s)ds\Big](x',x_n)\Big|dx'dx_n$$
$$\leq Cg(t)t^{1-\frac{\alpha}{2}},$$

which is (5.2). Note that $K(\frac{s}{t}) + \log_e\Big(1+\sqrt{\frac{t}{t-s}}\Big) \leq \log_e(1+\sqrt{2}\,)$ for each $0 \leq s \leq \frac{t}{2}$, $t > 0$, where the definition of $K(\lambda)$ is given in the proof of (5.7). Whence, similar to the proof of (5.7), we conclude for $t > 0$

$$\int_0^{\sqrt{t}}\int_{\mathbb{R}^{n-1}}\Big|\Big[\int_0^{\frac{t}{2}} e^{-(t-s)A}b(\cdot,s)ds\Big](x',x_n)\Big|dx'dx_n$$
$$\leq C\int_0^{\frac{t}{2}}(1+\log_e(\sqrt{t}+\sqrt{t-s}\,)-\log_e\sqrt{t-s}\,)^{1-\alpha}$$

$$\times \big((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} K(\frac{s}{t})\big)^{\alpha} \int_{\mathbb{R}^n_+} |y|^{\alpha} |b(y,s)| dy ds$$

$$\le C t^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \Big(1 + \log_e\Big(1 + \sqrt{\frac{t}{t-s}}\Big)\Big)^{1-\alpha}$$

$$\times \big(1 + K(\frac{s}{t})\big)^{\alpha} \| |y|^{\alpha} |b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)} ds$$

$$\le C t^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \| |y|^{\alpha} b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)} ds,$$

which is (5.1). □

Lemma 5.2. *(See [21,22].) Let $0 < \eta < 1$, $0 \le \alpha < 1$ and $1 \le k \le n$. Then for any $u \in C^{\infty}_{0,\sigma}(\mathbb{R}^n_+)$*

$$\Big\| \sum_{i,j=1}^{n} x_n^{-\eta} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \Big\|_{L^1(\mathbb{R}^n_+)} \le C_\eta (\|u\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n_+)});$$

$$\Big\| \sum_{i,j=1}^{n} \nabla \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \Big\|_{L^1(\mathbb{R}^n_+)} \le C (\|u\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|^2_{L^2(\mathbb{R}^n_+)});$$

and

$$\Big\| \sum_{i,j=1}^{n} |x|^{\alpha} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \Big\|_{L^1(\mathbb{R}^n_+)}$$
$$\le C (\|u\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n_+)} + \| |y|^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n_+)} + \| |y|^{\frac{\alpha}{2}} \nabla u\|^2_{L^2(\mathbb{R}^n_+)}).$$

Proof of Theorem 1.4. It follows from Theorem 1.3, Lemmata 2.1, 4.1, 5.1, 5.2 that for $0 < \alpha < 1$ and $t > 0$

$$\|u(t)\|_{L^1(\mathbb{R}^{n-1} \times (0,\sqrt{t}))}$$

$$\le \|e^{-tA} u_0\|_{L^1(\mathbb{R}^{n-1} \times (0,\sqrt{t}))} + \Big\| \int_0^{\frac{t}{2}} e^{-(t-s)A} P u(s) \cdot \nabla u(s) \Big\|_{L^1(\mathbb{R}^{n-1} \times (0,\sqrt{t}))} ds$$

$$+ \Big\| \int_{\frac{t}{2}}^{t} e^{-(t-s)A} P u(s) \cdot \nabla u(s) \Big\|_{L^1(\mathbb{R}^{n-1} \times (0,\sqrt{t}))} ds$$

$$\le C t^{-\frac{\alpha}{2}} \int_{\mathbb{R}^n_+} |y|^{\alpha} |u_0(y)| dy + C t^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \| |y|^{\alpha} P u(s) \cdot \nabla u(s) \|_{L^1(\mathbb{R}^n_+)} ds$$

$$
\begin{aligned}
&+ Ct^{1-\frac{\alpha}{2}} \sup_{\frac{t}{2}\le s\le t} \||y|^\alpha Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} \\
&\le Ct^{-\frac{\alpha}{2}} \int_{\mathbb{R}^n_+} (1+|y|)|u_0(y)|dy \\
&\quad + Ct^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \big(\||y|^\alpha u(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} + \big\| \sum_{i,j=1}^n |y|^\alpha \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\big\|_{L^1(\mathbb{R}^n_+)}\big)ds \\
&\quad + Ct^{1-\frac{\alpha}{2}} \sup_{\frac{t}{2}\le s\le t} \big(\||y|^\alpha u(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} + \big\| \sum_{i,j=1}^n |y|^\alpha \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\big\|_{L^1(\mathbb{R}^n_+)}\big) \\
&\le Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^\alpha u(s)\|_{L^2(\mathbb{R}^n_+)}\|u(s)\|_{L^2(\mathbb{R}^n_+)} + \||y|^\alpha \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\big)ds \\
&\quad + Ct^{1-\frac{\alpha}{2}} \sup_{\frac{t}{2}\le s\le t} \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^\alpha u(s)\|_{L^2(\mathbb{R}^n_+)}\|u(s)\|_{L^2(\mathbb{R}^n_+)} + \||y|^\alpha \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\big) \\
&\le Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} \big((1+s)^{-1-\frac{n}{2}} + (1+s)^{-\frac{n+1}{2}+\frac{\alpha}{2}}\big)ds \\
&\quad + Ct^{1-\frac{\alpha}{2}} \sup_{\frac{t}{2}\le s\le t} \big((1+s)^{-1-\frac{n}{2}} + (1+s)^{-\frac{n+1}{2}+\frac{\alpha}{2}}\big) \\
&\le C_\alpha t^{-\frac{\alpha}{2}};
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} &\le \|\nabla e^{-tA}u_0\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \|(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)\nabla e^{-(t-s)A}Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+))}ds \\
&\le Ct^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^n_+} |y|^\alpha |u_0(y)|dy \\
&\quad + C\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \||y|^\alpha Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\quad + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \||y|^\alpha Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds
\end{aligned}
$$

$$
\begin{aligned}
&\le Ct^{-\frac{1+\alpha}{2}} \int\limits_{\mathbb{R}^n_+} (1+|y|)|u_0(y)|dy \\
&\quad + Ct^{-\frac{1+\alpha}{2}} \int\limits_0^{\frac{t}{2}} \big(\||y|^\alpha u(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \big\| \sum_{i,j=1}^n |y|^\alpha \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\big\|_{L^1(\mathbb{R}^n_+)}\big) ds \\
&\quad + C\int\limits_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \big(\||y|^\alpha u(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \big\| \sum_{i,j=1}^n |y|^\alpha \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\big\|_{L^1(\mathbb{R}^n_+)}\big) ds \\
&\le Ct^{-\frac{1+\alpha}{2}} + Ct^{-\frac{1+\alpha}{2}} \int\limits_0^{\frac{t}{2}} \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^\alpha u(s)\|_{L^2(\mathbb{R}^n_+)}\|u(s)\|_{L^2(\mathbb{R}^n_+)} + \||y|^\alpha \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\big) ds \\
&\quad + C\int\limits_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^\alpha u(s)\|_{L^2(\mathbb{R}^n_+)}\|u(s)\|_{L^2(\mathbb{R}^n_+)} + \||y|^\alpha \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\big) ds \\
&\le Ct^{-\frac{1+\alpha}{2}} + Ct^{-\frac{1+\alpha}{2}} \int\limits_0^{\frac{t}{2}} \big((1+s)^{-1-\frac{n}{2}} + (1+s)^{-\frac{n+1}{2}+\frac{\alpha}{2}}\big) ds \\
&\quad + C\int\limits_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}} \big((1+s)^{-1-\frac{n}{2}} + (1+s)^{-\frac{n+1}{2}+\frac{\alpha}{2}}\big) ds \\
&\le C_\alpha t^{-\frac{1+\alpha}{2}}. \qquad (5.12)
\end{aligned}
$$

To proceed, let $0 < \eta < 1$. Then for $t > 0$

$$
\begin{aligned}
&\|x_n^{-\eta} u(t)\cdot\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \\
&\le \int\limits_0^1 \int\limits_{\mathbb{R}^{n-1}} x_n^{-\eta}|u(x,t)||\nabla u(x,t)|dx'dx_n + \|u(t)\cdot\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \\
&\le \|x_n^{-\eta}u(t)\|_{L^2(\mathbb{R}^{n-1}\times(0,1))}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)} + \|u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}. \qquad (5.13)
\end{aligned}
$$

One-dimensional Hardy inequality yields for $t > 0$

$$\begin{aligned}\|x_n^{-\eta}u(t)\|^2_{L^2(\mathbb{R}^{n-1}\times(0,1))} &\le \int\limits_{\mathbb{R}^{n-1}}\int\limits_0^1 x_n^{2-2\eta}\frac{|u(x',x_n,t)|^2}{x_n^2}dx_n dx' \\ &\le \int\limits_{\mathbb{R}^{n-1}}\int\limits_0^\infty \frac{|u(x',x_n,t)|^2}{x_n^2}dx_n dx' \\ &\le C\int\limits_{\mathbb{R}^{n-1}}\int\limits_0^\infty \big|\partial_n u(x',x_n,t)\big|^2 dx_n dx' \\ &\le C\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

Whence from (5.13), we obtain for $0 < \eta < 1$ and $t > 0$

$$\|x_n^{-\eta}u(t)\cdot\nabla u(t)\|_{L^1(\mathbb{R}^n_+)} \le C(\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(t)\|^2_{L^2(\mathbb{R}^n_+)}). \tag{5.14}$$

Before giving the proof of $\|\nabla^2 u(t)\|_{L^1(\mathbb{R}^n_+)}$, we recall a very useful estimate (see [25]): Let $f = (f_1, f_2, \cdots, f_n) \in W^{1,1}(\mathbb{R}^n_+)$ $(n \ge 2)$ satisfy $\nabla\cdot f = 0$ in $\mathbb{R}^n_+$ and $f_n\,|_{\partial\mathbb{R}^n_+} = 0$. Then for any $0 < \eta < 1$ and $t > 0$

$$\|\nabla^2 e^{-tA}f\|_{L^1(\mathbb{R}^n_+)} \le C\big(t^{-\frac{1}{2}}\|\nabla f\|_{L^1(\mathbb{R}^n_+)} + t^{-1+\frac{\eta}{2}}\int\limits_{\mathbb{R}^n_+} y_n^{-\eta}|f(y)|dy\big). \tag{5.15}$$

In addition, let $g = (g_1, g_2, \cdots, g_n) \in W^{1,1}_{0,\sigma}(\mathbb{R}^n_+)$ $(n \ge 2)$, checking the proofs of (A.5)–(A.7) in [25], we find for $t > 0$

$$\|\nabla^2 e^{-tA}g\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-\frac{1}{2}}\|\nabla g\|_{L^1(\mathbb{R}^n_+)}. \tag{5.16}$$

Whence, from (5.12), (5.14)–(5.16) and Lemmata 2.1, 5.2, we deduce for $0 < \alpha < 1$ and $t > 1$

$$\begin{aligned}\|\nabla^2 u(t)\|_{L^1(\mathbb{R}^n_+)} &\le \|\nabla^2 e^{-\frac{t}{2}A}u(\frac{t}{2})\|_{L^1(\mathbb{R}^n_+)} + \int\limits_{\frac{t}{2}}^t \|\nabla^2 e^{-(t-s)A}Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\ &\le Ct^{-\frac{1}{2}}\|\nabla u(\frac{t}{2})\|_{L^1(\mathbb{R}^n_+)} + C\int\limits_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}\|\nabla Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\ &\quad + C\int\limits_{\frac{t}{2}}^t (t-s)^{-1+\frac{\eta}{2}}\|x_n^{-\eta}Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds\end{aligned}$$

$$
\begin{aligned}
&\le Ct^{-1-\frac{\alpha}{2}} + C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} \big(\|\nabla\big(u(s)\cdot\nabla u(s)\big)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \big\|\nabla\big(\sum_{i,j=1}^{n} \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\big)\big\|_{L^1(\mathbb{R}^n_+)}\big)ds \\
&\quad + C\int_{\frac{t}{2}}^{t} (t-s)^{-1+\frac{\eta}{2}} \big(\|x_n^{-\eta}u(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \|\sum_{i,j=1}^{n} x_n^{-\eta}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\|_{L^1(\mathbb{R}^n_+)}\big)ds \\
&\le Ct^{-1-\frac{\alpha}{2}} + C\int_{\frac{t}{2}}^{t} \big((t-s)^{-\frac{1}{2}} + (t-s)^{-1+\frac{\eta}{2}}\big) \\
&\quad \times \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}\big)ds \\
&\le Ct^{-1-\frac{\alpha}{2}} + C\int_{\frac{t}{2}}^{t} \big((t-s)^{-\frac{1}{2}} + (t-s)^{-1+\frac{\eta}{2}}\big) \\
&\quad \times \big((1+s)^{-1-\frac{n}{2}} + (1+s)^{-2-\frac{n}{2}} + s^{-3-\frac{n}{2}}\big)ds \quad \text{where} \quad 0<\eta<1 \\
&\le C_\alpha t^{-1-\frac{\alpha}{2}}. \qquad \square
\end{aligned}
$$

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