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Diffusive wave in the low Mach limit for compressible Navier–Stokes equations

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ABSTRACT

The low Mach limit for 1D non-isentropic compressible Navier–Stokes flow, whose density and temperature have different asymptotic states at infinity, is rigorously justified. The problems are considered on both well-prepared and ill-prepared data. For the well-prepared data, the solutions of compressible Navier–Stokes equations are shown to converge to a nonlinear diffusion wave solution globally in time as Mach number goes to zero when the difference between the states at $\pm\infty$ is suitably small. In particular, the velocity of diffusion wave is only driven by the variation of temperature. It is further shown that the solution of compressible Navier–Stokes system also has the same property when Mach number is small, which has never been observed before. The convergence rates on both Mach number and time are also obtained for the well-prepared data. For the ill-prepared data, the limit relies on the uniform estimates including weighted time derivatives and an extended convergence lemma. And the difference between the states at $\pm\infty$ can be arbitrary large in this case.

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Contents

1. Introduction

The non-isentropic Navier–Stokes system in $\mathbb{R}^n$ is as follows

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}(2\mu D(u)) + \nabla(\lambda \operatorname{div} u), \\ \partial_t(\rho(e + \frac{1}{2}|u|^2)) + \operatorname{div}(\rho u(e + \frac{1}{2}|u|^2) + Pu) = \operatorname{div}(\kappa\nabla\mathcal{T}) + \operatorname{div}(2\mu D(u)u + \lambda \operatorname{div} u u), \end{cases} \tag{1.1}$$

for $t > 0, x \in \mathbb{R}^n$. Here the unknown functions ρ, u, and $\mathcal{T}$ represent the density, velocity, and temperature, respectively. The pressure function and internal function are defined by

$$P = R\rho\mathcal{T}, \quad e = c_v\mathcal{T}, \tag{1.2}$$

where the parameters $R > 0$ and $c_v > 0$ are the gas constant and the heat capacity at constant volume, respectively. $D(u)$ is the deformation tensor given by

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^t),$$

where $(\nabla u)^t$ denote the transpose of matrix ∇u. μ and λ are the Lamé viscosity coefficients which satisfy

$$\mu > 0, \quad 2\mu + n\lambda > 0,$$

and $\kappa > 0$ is the heat conductivity coefficient. For simplicity, we assume that μ, λ, κ are constants.

It is noted $c_v = (\gamma - 1)/R$ where $\gamma > 1$ is the adiabatic exponent. In this paper, since we consider the low Mach limit for smooth solutions of non-isentropic compressible Navier–Stokes equation, we normalize $R = 1$ and $c_v = 1$ for simplicity of presentation, see also [25]. And we point out that our results and energy estimates in this paper still hold for any given $c_v > 0$ (equivalently for any given $\gamma > 1$), the only difference is that some constants of this paper may depend on c_v.

Let ε be the compressibility parameter which is a nondimensional quantity. As in [39], we set

$$t \to \varepsilon t, \quad x \to x, \quad u \to \varepsilon u, \quad \mu \to \varepsilon\mu, \quad \lambda \to \varepsilon\lambda, \quad \kappa \to \varepsilon\kappa. \tag{1.3}$$

Based on the above changes of variables, the compressible Navier–Stokes system (1.1), written after the nondimensionalization, becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{\nabla P^\varepsilon}{\varepsilon^2} = \mathrm{div}(2\mu D(u^\varepsilon)) + \nabla(\lambda \mathrm{div} u^\varepsilon), \\ \partial_t(\rho^\varepsilon \mathcal{T}^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \mathcal{T}^\varepsilon) + P^\varepsilon \mathrm{div} u^\varepsilon = \mathrm{div}(\kappa \nabla \mathcal{T}^\varepsilon) + \varepsilon^2[2\mu|D(u^\varepsilon)|^2 + \lambda|\mathrm{div} u^\varepsilon|^2], \end{cases} \tag{1.4}$$

in which the typical mean fluid velocity has been chosen as the ratio of time units to space units. Accordingly, the parameter ε essentially presents the maximum Mach number of the fluid. The limit of solutions of (1.4) as ε goes to 0 is usually called as low Mach limit [29,30]. The purpose of the low Mach number approximation is to justify that the compression, due to pressure variations, can be neglected. This is a common assumption when discussing the fluid dynamics of highly subsonic flows. Similar to [2,39], we assume that the pressure is a small perturbation of a given constant state $\underline{P} > 0$, *i.e.*

$$P^\varepsilon = \underline{P} + O(\varepsilon), \tag{1.5}$$

which satisfies the following equation

$$\partial_t(P^\varepsilon) + \mathrm{div}(u^\varepsilon P^\varepsilon) + P^\varepsilon \mathrm{div} u^\varepsilon = \mathrm{div}(\kappa \nabla \mathcal{T}^\varepsilon) + \varepsilon^2[2\mu|D(u^\varepsilon)|^2 + \lambda|\mathrm{div} u^\varepsilon|^2]. \tag{1.6}$$

Formally, as the Mach number tends to zero, the equations (1.6), $(1.4)_2$ and $(1.4)_3$ become

$$\begin{cases} \mathrm{div}(2\underline{P}u - \kappa\nabla\mathcal{T}) = 0, \\ \rho(u_t + u \cdot \nabla u) + \nabla\pi = \mathrm{div}(2\mu D(u)) + \nabla(\lambda \mathrm{div} u), \\ \rho(\mathcal{T}_t + u \cdot \nabla\mathcal{T}) + \underline{P}\mathrm{div} u = \mathrm{div}(\kappa\nabla\mathcal{T}), \end{cases} \tag{1.7}$$

and the density satisfies $\rho = \underline{P}/\mathcal{T}$ and

$$\rho_t + \mathrm{div}(\rho u) = 0. \tag{1.8}$$

Without loss of generality, we normalize $\underline{P}$ to be 1. Let the approximate initial data $(p^\varepsilon_{in}, u^\varepsilon_{in}, \mathcal{T}^\varepsilon_{in})$ converge to $(p_{in}, u_{in}, \mathcal{T}_{in})$ as $\varepsilon \to 0$ in some sense. Then, the approximate initial data is regarded as well-prepared data if u_{in} and $\mathcal{T}_{in}$ satisfies (1.7)$_1$. Otherwise, it is called as ill-prepared data.

The low Mach limit is an important and interesting problem in the fluid dynamics. There have been a lot of literatures on the low Mach limit. The first result is due to Klainerman and Majda [29,30], in which they proved the incompressible limit of the isentropic Euler equations to the incompressible Euler equations for local smooth solutions with well-prepared data. By using the fast decay of acoustic waves, Ukai [42] verified the low Mach limit for the general data, see also [3]. The half plane and exterior domain cases were considered in [19–21]. In [37], Schochet showed the limit of the full compressible Euler equations to the incompressible inhomogeneous Euler equations in bounded domain for local smooth solutions with well-prepared initial data. He [38] further studied the fast singular limit for general hyperbolic partial differential equations. The major breakthrough on the ill-prepared data is due to Métivier and Schochet [35], in which they proved the low Mach limit of full Euler equations in the whole space by a significant convergence lemma on acoustic waves. Later, Alazard [1] extended [35] to the exterior domain. The one dimension spatial periodic case on the full compressible Euler equations was considered in [36]. For other interesting works, see [6] and the references therein.

For the isentropic Navier–Stokes equations, the low Mach limit of the global weak solutions with general initial data have been well studied under various boundary conditions, see [9,10,32,33]. For the non-isentropic Navier–Stokes equations, Alazard [2] justified the low Mach limit in the whole space for the ill-prepared data, by employing a uniform estimate and the convergence lemma of [35]. For the bounded domain, the low Mach limit was justified by Jiang–Ou [26] and Dou–Jiang–Ou [11]. A dispersive Navier–Stokes system was also studied in [31]. For other interesting works, see [7,8,14,27,34,39] for Navier–Stokes equations, [13,15,22–25] for MHD equations and the references therein.

Note that in the whole space, all results above require that both density and temperature have the constant background, *i.e.*

$$\rho^\varepsilon(x,t) \to \underline{\rho}, \text{ and } \mathcal{T}^\varepsilon(x,t) \to \underline{\mathcal{T}}, \quad \text{as} \quad |x| \to \infty, \tag{1.9}$$

where $\underline{\rho} > 0$ and $\underline{\mathcal{T}} > 0$ are given constants. In this paper, we will study the low Mach limit when the background is not constant state in the one dimensional case, that is

$$(\rho^\varepsilon, \mathcal{T}^\varepsilon)(x,t) \to (\rho_\pm, \mathcal{T}_\pm), \quad \text{as } x \to \pm\infty, \quad \text{with} \quad \rho_-\mathcal{T}_- = \rho_+\mathcal{T}_+, \tag{1.10}$$

where $\mathcal{T}_-$ may not be equal to $\mathcal{T}_+$, and want to know what happens in the limiting process. In fact, we find the solutions of compressible Navier–Stokes equations converge to a nonlinear diffusion wave solution globally in time as Mach number goes to zero. In particular, the velocity of diffusion wave is only driven by the variation of temperature. This phenomenon looks like thermal creep flow [16,18,28,41]. Moreover, when Mach number is small, the compressible Navier–Stokes system also has the same property, which has never been observed before.

Now we begin to formulate the main results. The system (1.4) and the limiting system (1.7) in one dimension become, respectively,

$$\begin{cases} \partial_t \rho^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x = 0, \\ \rho^\varepsilon (u_t^\varepsilon + u^\varepsilon u_x^\varepsilon) + \frac{P_x^\varepsilon}{\varepsilon^2} = \tilde{\mu} u_{xx}^\varepsilon, \\ \rho^\varepsilon (\mathcal{T}_t^\varepsilon + u^\varepsilon \mathcal{T}_x^\varepsilon) + P^\varepsilon u_x^\varepsilon = \kappa \mathcal{T}_{xx}^\varepsilon + \tilde{\mu}\varepsilon^2 |u_x^\varepsilon|^2, \end{cases} \tag{1.11}$$

and

$$\begin{cases} (2u - \kappa \mathcal{T}_x)_x = 0, \quad \rho = \mathcal{T}^{-1}, \\ \rho(u_t + u u_x) + \pi_x = \tilde{\mu} u_{xx}, \\ \rho(\mathcal{T}_t + u\mathcal{T}_x) + u_x = \kappa \mathcal{T}_{xx}, \end{cases} \tag{1.12}$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and $\tilde{\mu} = 2\mu + \lambda > 0$. We shall construct a special solution of (1.12). Indeed, from $(1.12)_1$, we choose

$$u = \frac{\kappa}{2}\mathcal{T}_x = -\frac{\kappa}{2}\frac{\rho_x}{\rho^2}. \tag{1.13}$$

Substituting (1.13) into (1.12), one obtains the following nonlinear diffusion equation

$$\rho_t = \left(\frac{\kappa}{2}\frac{\rho_x}{\rho}\right)_x. \tag{1.14}$$

We consider the boundary condition at the far field, *i.e.*, $\lim_{x\to\pm\infty} \rho(x,t) = \rho_\pm$. From [4] and [12], it is known that the nonlinear diffusion equation (1.14) admits a unique self-similar solution $\Xi(\eta)$, $\eta = \frac{x}{\sqrt{1+t}}$ satisfying $\Xi(\pm\infty, t) = \rho_\pm$. Let $\delta = |\rho_+ - \rho_-|$, then $\Xi(t, x)$ satisfies

$$\Xi_x(t,x) = \frac{O(1)\delta}{\sqrt{(1+t)}} e^{-\frac{x^2}{4d(\rho_\pm)(1+t)}}, \quad \text{as} \quad x \to \pm\infty, \quad \text{where} \quad d(\rho) = \frac{\kappa}{2\rho}.$$

We define

$$(\bar{\rho}, \bar{u}, \bar{\mathcal{T}}) \doteq \left(\Xi, \ -\frac{\kappa}{2}\frac{\Xi_x}{\Xi^2}, \ \Xi^{-1}\right), \tag{1.15}$$

which is a special solution of (1.12), that is

$$\begin{cases} (2\bar{u} - \kappa \bar{\mathcal{T}}_x)_x = 0, \ \bar{\rho} = \bar{\mathcal{T}}^{-1}, \\ \bar{\rho}(\bar{u}_t + \bar{u}\bar{u}_x) + \bar{\pi}_x = \tilde{\mu}\bar{u}_{xx}, \\ \bar{\rho}(\bar{\mathcal{T}}_t + \bar{u}\bar{\mathcal{T}}_x) + \bar{u}_x = \kappa \bar{\mathcal{T}}_{xx}, \end{cases} \tag{1.16}$$

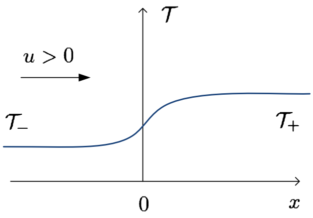

Fig. 1. $0 < \mathcal{T}_- < \mathcal{T}_+$.

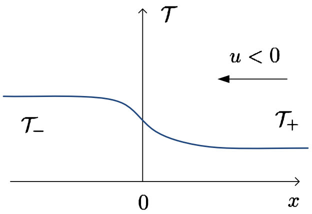

Fig. 2. $\mathcal{T}_- > \mathcal{T}_+ > 0$.

with $\bar{\pi} = \tilde{\mu}\bar{u}_x - \bar{\rho}\bar{u}^2 + \frac{\kappa}{2}\frac{\bar{\rho}_t}{\bar{\rho}}$. In particular, $\bar{u} = \frac{\kappa}{2}\bar{\mathcal{T}}_x$ indicates that the flow is driven by the variation of temperature. In other word, the flow moves along the direction from the low temperature to the high one, see Figs. 1 and 2, and thus behaves like thermal creep flow. Indeed, Slemrod [41] mentioned: "*This leaves open the issue of finding the correct extended fluid dynamics for moderate dense gas or polyatomic gas*", *i.e.*, how to construct a thermal creep flow for moderate dense gas or polyatomic gas is still an open problem. In this paper, for well-prepared data, we will show that the solution of compressible Navier–Stokes equation behaviors as thermal creep flow in some sense when the Mach number is small. It should be noted that the usual Poiseuille flow moves from the high pressure part to the low one due to the difference of pressure. We will state our main results in the following two subsections.

1.1. Main results for well-prepared data

In this subsection, we consider the low Mach limit around the diffusive wave with well-prepared initial data. It is more convenient to use the *Lagrangian* coordinates for the global behavior of solutions. That is, take the coordinate transformation

$$(x,t) \to \Big(\int_{(0,0)}^{(x,t)} \rho^\varepsilon(z,s)dz - (\rho^\varepsilon u^\varepsilon)(z,s)ds, s\Big),$$

which is still denoted as (x,t) without confusion. It is noted that the above transformation is independent of the path of integration due to the continuity equation $(1.11)_1$. We also point out that the Eulerian and Lagrangian formulations are equivalent since we consider the smooth solution of compressible Navier–Stokes equations, see also [5,40]. Let $v = \frac{1}{\rho}$

be the specific volume. Then, in Lagrangian coordinates, the system (1.11) can be written equivalently as

$$\begin{cases} v_t^\varepsilon - u_x^\varepsilon = 0, \\ u_t^\varepsilon + \frac{1}{\varepsilon^2} P_x^\varepsilon = \tilde{\mu}(\frac{u_x^\varepsilon}{v^\varepsilon})_x, \\ (\mathcal{T}^\varepsilon + \frac{1}{2}|\varepsilon u^\varepsilon|^2)_t + (P^\varepsilon u^\varepsilon)_x = \kappa(\frac{1}{v^\varepsilon}\mathcal{T}_x^\varepsilon)_x + \varepsilon^2(\frac{\tilde{\mu}}{v^\varepsilon} u^\varepsilon u_x^\varepsilon)_x, \end{cases} \tag{1.17}$$

where the pressure $P = \mathcal{T}/v$. Similarly, the limiting system (1.16) in the Lagrangian coordinate becomes

$$\begin{cases} (2u - \frac{\kappa}{v}\mathcal{T}_x)_x = 0, \ v = \mathcal{T}, \\ u_t + \pi_x = \tilde{\mu}(\frac{1}{v} u_x)_x, \\ \mathcal{T}_t + u_x = \kappa(\frac{1}{v}\mathcal{T}_x)_x. \end{cases} \tag{1.18}$$

As in (1.15), we can construct a special diffusive wave solution $(\bar{v}, \bar{u}, \bar{\mathcal{T}}, \bar{\pi})$ of (1.18) by choosing

$$(\bar{v}, \bar{u}, \bar{\mathcal{T}}) := (\hat{\mathcal{T}}, \frac{\kappa \hat{\mathcal{T}}_x}{2\hat{\mathcal{T}}}, \hat{\mathcal{T}}), \tag{1.19}$$

where $\hat{\mathcal{T}}(\eta), \eta = \frac{x}{\sqrt{1+t}}$ is the unique self-similar solution of the following diffusion equation

$$\mathcal{T}_t = \left(\frac{\kappa \mathcal{T}_x}{2\mathcal{T}}\right)_x, \ \text{with} \ \lim_{x\to\pm\infty} \mathcal{T} = \mathcal{T}_\pm, \tag{1.20}$$

and $\bar{\pi}$ can be solved by $(1.18)_2$. Let $\delta = |\mathcal{T}_+ - \mathcal{T}_-|$, then $\hat{\mathcal{T}}(t, x)$ satisfies

$$\hat{\mathcal{T}}_x(t, x) = O(1)\delta(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4d(\mathcal{T}_\pm)(1+t)}}, \ \ d(\mathcal{T}) = \frac{\kappa}{2\mathcal{T}}, \ \ \text{as} \ \ x \to \pm\infty. \tag{1.21}$$

Indeed, $(\bar{v}, \bar{u}, \bar{\mathcal{T}})$ is the Lagrangian version of diffusion wave solution (1.15).

Since $(\bar{v}, \bar{u}, \bar{\mathcal{T}})$ is not the solution of (1.17) and some non-integrated error terms with respect to time t and slow ε-decay terms should appear in the low Mach limiting analysis, we need to introduce a new profile to approximate the system (1.17), *i.e.*,

$$(\tilde{v}, \tilde{u}, \tilde{\mathcal{T}}) := (\bar{v}, \bar{u}, \bar{\mathcal{T}} - \frac{1}{2}|\varepsilon \bar{u}|^2). \tag{1.22}$$

Then a direct calculation gives that

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ \varepsilon^2 \tilde{u}_t + \tilde{P}_x = \tilde{\mu}\varepsilon^2(\frac{\tilde{u}_x}{\tilde{v}})_x + \tilde{R}_{1x}, \\ \left(\tilde{\mathcal{T}} + \frac{1}{2}|\varepsilon \tilde{u}|^2\right)_t + (\tilde{P}\tilde{u})_x = (\kappa\frac{\tilde{\mathcal{T}}_x}{\tilde{v}})_x + \varepsilon^2(\frac{\tilde{\mu}}{\tilde{v}}\tilde{u}\tilde{u}_x)_x + \tilde{R}_{2x}, \end{cases} \tag{1.23}$$

where

$$\tilde{R}_1=\varepsilon^2\frac{\kappa\hat{\mathcal{T}}_t}{2\hat{\mathcal{T}}}-\varepsilon^2\frac{\tilde{u}^2}{2\tilde{v}}-\varepsilon^2\frac{\tilde{\mu}}{\tilde{v}}\tilde{u}_x=O(1)\delta\varepsilon^2(1+t)^{-1}e^{-\frac{c_\pm x^2}{1+t}},\quad \text{as}\quad |x|\to\infty, \tag{1.24}$$

$$\tilde{R}_2=\varepsilon^2\frac{\kappa}{\hat{\mathcal{T}}}\tilde{u}\tilde{u}_x-\frac{\varepsilon^2}{2\tilde{v}}\tilde{u}^3-\frac{\varepsilon^2}{\tilde{v}}\tilde{\mu}\tilde{u}\tilde{u}_x=O(1)\delta\varepsilon^2(1+t)^{-\frac{3}{2}}e^{-\frac{c_\pm x^2}{1+t}},\quad \text{as}\quad |x|\to\infty. \tag{1.25}$$

It is noted that the system (1.23) approximate the original compressible Navier–Stokes equations (1.17) very well, up to error terms $\tilde{R}_{1x}$ and $\tilde{R}_{2x}$, when ε is sufficiently small. Moreover, since we will integrate the differences between the solutions of original system (1.17) and the approximate system (1.23) with respect to space variable in (2.3) below, so it is very important that the error terms are in the form of x-derivatives. From (1.19) and (1.22), we also note that

$$\|(\bar{v}-\tilde{v},\bar{u}-\tilde{u},\bar{\mathcal{T}}-\tilde{\mathcal{T}})(t)\|\le C\varepsilon^2(1+t)^{-1}, \tag{1.26}$$

which implies that $(\tilde{v},\tilde{u},\tilde{\mathcal{T}})$ approximate the diffusive wave solution $(\bar{v},\bar{u},\bar{\mathcal{T}})$ very well when ε is small.

Now we supplement the system (1.17) with the initial data

$$(v^\varepsilon,u^\varepsilon,\mathcal{T}^\varepsilon)|_{t=0}=(\tilde{v},\tilde{u},\tilde{\mathcal{T}})(x,0), \tag{1.27}$$

then we have the following global existence and uniform estimates.

Theorem 1.1 *(Uniform estimates for well-prepared data). Let $(\tilde{v},\tilde{u},\tilde{\mathcal{T}})(x,t)$ be the diffusive wave defined in (1.22) with the wave strength $\delta=|\mathcal{T}_+-\mathcal{T}_-|$. There exist positive constants δ_0 and ε_0, such that if $\delta\le\delta_0$ and $\varepsilon\le\varepsilon_0$, then the Cauchy problem (1.17), (1.27) has a unique global smooth solution $(v^\varepsilon,u^\varepsilon,\mathcal{T}^\varepsilon)$ satisfying*

$$\begin{cases}\|(v^\varepsilon-\tilde{v},\varepsilon u^\varepsilon-\varepsilon\tilde{u},\mathcal{T}^\varepsilon-\tilde{\mathcal{T}})(t)\|^2_{L^2_x}\le C\sqrt{\delta}\varepsilon^3(1+t)^{-1+C_0\sqrt{\delta}},\\ \|(v^\varepsilon-\tilde{v},\varepsilon u^\varepsilon-\varepsilon\tilde{u},\mathcal{T}^\varepsilon-\tilde{\mathcal{T}})_x(t)\|^2_{L^2_x}\le C\sqrt{\delta}\varepsilon^2(1+t)^{-\frac{3}{2}+C_0\sqrt{\delta}},\end{cases} \tag{1.28}$$

where C and C_0 are positive constants independent of ε and δ.

Based on the uniform estimates (1.28) and Sobolev embedding, we justify the following low Mach limit.

Corollary 1.2 *(Low Mach limit for well-prepared data). Under the assumptions of Theorem 1.1, it follows from (1.26) and (1.28) that as $\varepsilon\to 0$,*

$$\begin{cases}\|(v^\varepsilon-\bar{v},\mathcal{T}^\varepsilon-\bar{\mathcal{T}})(t)\|_{L^\infty(\mathbb{R})}\le C\delta^{\frac{1}{4}}\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{1}{2}}\to 0,\\ \|(u^\varepsilon-\bar{u})(t)\|_{L^\infty(\mathbb{R})}\le C\delta^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{1}{2}}\to 0.\end{cases} \tag{1.29}$$

Note that the velocity $\bar{u}$ is driven by the variation of temperature $\tilde{\mathcal{T}}$, *i.e.*, $\bar{u} = \frac{\kappa \hat{\mathcal{T}}_x}{2\hat{\mathcal{T}}}$. Without loss of generality, we assume that $\mathcal{T}_- < \mathcal{T}_+$, then for any given positive constant $\eta_0 > 0$, there exists $c_{\eta_0} > 0$ such that

$$\hat{\mathcal{T}}'(\eta) > c_{\eta_0}\delta > 0, \quad \text{for} \quad |\eta| \le \eta_0. \tag{1.30}$$

Moreover, we have

Corollary 1.3 *(Driven by the variation of temperature). For the solutions obtained in Theorem 1.1, there exists a small positive constant $\varepsilon_1 = \varepsilon_1(\eta_0) \le \varepsilon_0$, such that if $\varepsilon \le \varepsilon_1$, it follows from (1.29) and (1.30) that*

$$\begin{cases} 0 < \frac{c_{\eta_0}\delta}{C_1\sqrt{1+t}} < \frac{1}{C_1}\hat{\mathcal{T}}_x \le u^\varepsilon(x,t) \le C_1\hat{\mathcal{T}}_x, \\ \qquad 0 < \frac{1}{2}\hat{\mathcal{T}}_x \le \mathcal{T}^\varepsilon_x(x,t) \le \frac{3}{2}\hat{\mathcal{T}}_x, \end{cases} \quad \text{for } |x| \le \eta_0(1+t)^{\frac{1}{2}},\ t \ge 0, \tag{1.31}$$

which yields immediately that

$$\frac{2}{3C_1}\mathcal{T}^\varepsilon_x(x,t) \le u^\varepsilon(x,t) \le 2C_1\mathcal{T}^\varepsilon_x(x,t), \ \text{for } |x| \le \eta_0(1+t)^{\frac{1}{2}},\ t \ge 0, \tag{1.32}$$

where C_1 is a suitably large positive constant depending only on $\mathcal{T}_\pm$.

Remark 1.4. The estimate (1.32) shows that the velocity u^ε of the compressible Navier–Stokes system (1.17) is also driven by the variation of temperature when ε is small. Note that the pressure P is almost 1, this phenomenon behaviors like thermal creep flow [16,28,41]. So, our result can be regarded as an answer to the open question of [41] in some sense.

Remark 1.5. All the above results for the well-prepared data hold globally in time.

1.2. Main results for ill-prepared data

To understand the role of the thermodynamics, as in [2], we rewrite the equations (1.4) by the pressure fluctuations p^ε, and temperature fluctuations θ^ε with

$$P^\varepsilon(x,t) = e^{\varepsilon p^\varepsilon(x,t)}, \quad \mathcal{T}^\varepsilon(x,t) = e^{\theta^\varepsilon(x,t)}. \tag{1.33}$$

It follows from (1.2) and (1.33) that

$$\rho^\varepsilon(x,t) = e^{\varepsilon p^\varepsilon(x,t) - \theta^\varepsilon(x,t)}. \tag{1.34}$$

Under these changes of variables, the asymptotic states at infinity are, as $x \to \pm\infty$

$$p^\varepsilon \to 0, \quad u^\varepsilon \to 0, \quad \text{and} \quad \theta^\varepsilon \to \theta_\pm, \ \text{as } x \to \pm\infty, \quad \text{where} \quad \theta_\pm = \ln \mathcal{T}_\pm. \tag{1.35}$$

The compressible Navier–Stokes system (1.11) takes the following equivalent form

$$\begin{cases} p_t^\varepsilon + u^\varepsilon \cdot p_x^\varepsilon + \frac{1}{\varepsilon}(2u^\varepsilon - \kappa e^{-\varepsilon p^\varepsilon + \theta^\varepsilon}\theta_x^\varepsilon)_x = \tilde{\mu}\varepsilon e^{-\varepsilon p^\varepsilon}|u_x^\varepsilon|^2 + \kappa e^{-\varepsilon p^\varepsilon + \theta^\varepsilon} p_x^\varepsilon \cdot \theta_x^\varepsilon, \\ e^{-\theta^\varepsilon}[u_t^\varepsilon + u^\varepsilon \cdot u_x^\varepsilon] + \frac{1}{\varepsilon} p_x^\varepsilon = \tilde{\mu} e^{-\varepsilon p^\varepsilon} u_{xx}^\varepsilon, \\ \theta_t^\varepsilon + u^\varepsilon \theta_x^\varepsilon + u_x^\varepsilon = \kappa e^{-\varepsilon p^\varepsilon}(e^{\theta^\varepsilon}\theta_x^\varepsilon)_x + \tilde{\mu}\varepsilon^2 e^{-\varepsilon p^\varepsilon}|u_x^\varepsilon|^2, \end{cases} \tag{1.36}$$

and the limiting system (1.12) becomes

$$\begin{cases} (2u - \kappa e^\theta \theta_x)_x = 0, \\ e^{-\theta}(u_t + u u_x) + \pi_x = (\tilde{\mu} u_x)_x, \\ \theta_t + u\theta_x + u_x = (\kappa e^\theta \theta_x)_x. \end{cases} \tag{1.37}$$

Since θ_- may not be equal to θ_+, we need to introduce a background profile $\tilde{\theta}$ for θ^ε. Here we choose

$$\tilde{\theta} := -\ln \Xi, \tag{1.38}$$

satisfying $\tilde{\theta} \to \theta_\pm$ as $x \to \pm\infty$. Then we define the following solution space: for $s \geq 4$,

$$\begin{aligned} \mathcal{N}(t) &:= \|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \tilde{\theta})(t)\|_{H^{s,\varepsilon}}^2 \\ &= \sum_{|\alpha|=0}^{s} \|\partial^\alpha (p^\varepsilon, u^\varepsilon)(t)\|_{L^2}^2 + \sum_{|\alpha|=0}^{s+1} \|\partial^\alpha (\varepsilon p^\varepsilon, \varepsilon u^\varepsilon)(t)\|_{L^2}^2 + \|(\theta^\varepsilon - \tilde{\theta})(t)\|_{L^2}^2 \\ &\quad + \sum_{|\alpha|=1}^{s+1} \|\partial^\alpha \theta^\varepsilon(t)\|_{L^2}^2 + \sum_{|\alpha|=0}^{s} \int_0^t \|\partial^\alpha (p_x^\varepsilon, u_x^\varepsilon)(\tau)\|_{L^2}^2 d\tau \\ &\quad + \sum_{|\alpha|=0}^{s+1} \int_0^t \|\partial^\alpha (\varepsilon u_x^\varepsilon, \theta_x^\varepsilon)(\tau)\|_{L^2}^2 d\tau, \end{aligned} \tag{1.39}$$

where $\partial^\alpha := (\varepsilon\partial_t)^{\alpha_0}\partial_x^{\alpha_1}$ with multi-index $\alpha = (\alpha_0, \alpha_1)$.

We supplement the system (1.36) with the following initial data

$$(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)|_{t=0} = (p_{in}^\varepsilon, u_{in}^\varepsilon, \theta_{in}^\varepsilon), \tag{1.40}$$

with

$$0 < \underline{a} \leq e^{-\varepsilon p_{in}^\varepsilon} \leq \bar{a}, \quad \text{and} \quad 0 < \underline{b} \leq e^{\theta_{in}^\varepsilon} \leq \bar{b},$$

where $\underline{a}, \bar{a}, \underline{b}$ and $\bar{b}$ are given constants independent of ε. Then the uniform estimates for (1.36) are:

Theorem 1.6 *(Uniform estimate for ill-prepared data). Let $s \geq 4$ be an integer, and assume that the initial data $(p^\varepsilon_{in}, u^\varepsilon_{in}, \theta^\varepsilon_{in})$ satisfy*

$$\|(p^\varepsilon_{in}, u^\varepsilon_{in}, \theta^\varepsilon_{in} - \tilde{\theta})\|^2_{H^{s,\varepsilon}} \leq \hat{C}_0 < \infty, \quad \text{for } \varepsilon \in (0,1], \tag{1.41}$$

where $\hat{C}_0$ is independent of ε. Then there exist positive constants T_0 and ε_0 depending only on $\hat{C}_0$ and $|\theta_+ - \theta_-|$ such that the Cauchy problem (1.36), (1.40) has a unique smooth solution $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ satisfying

$$\|(p^\varepsilon, u^\varepsilon, \theta^\varepsilon - \tilde{\theta})(t)\|^2_{H^{s,\varepsilon}} \leq \tilde{C}_0, \quad \text{for } t \in [0, T_0],\ \varepsilon \in (0, \varepsilon_0], \tag{1.42}$$

where the constant $\tilde{C}_0 > 0$ depends only on $\hat{C}_0$ and $|\theta_+ - \theta_-|$.

Remark 1.7. The time derivatives of the initial data in (1.41) are defined through the system (1.36).

Remark 1.8. The wave strength $|\theta_+ - \theta_-|$ (or equivalently $|\rho_+ - \rho_-|$) is allowed to be large. But the constants T_0 and ε_0 may depend on the wave strength.

Then we have the following low Mach limit.

Theorem 1.9 *(Low Mach limit for ill-prepared data). Under the assumptions of Theorem 1.6 and further assume that the initial data satisfy*

$$(p^\varepsilon_{in}, u^\varepsilon_{in}, \theta^\varepsilon_{in} - \tilde{\theta}) \to (p_{in}, u_{in}, \theta_{in} - \tilde{\theta}), \quad \text{in } H^s(\mathbb{R}) \text{ as } \varepsilon \to 0, \tag{1.43}$$

$$|\theta_{in}(x) - \theta_+| \leq Cx^{-1-\sigma}, \text{ for } x \in [1, +\infty), \tag{1.44}$$

where σ and C are some positive constants. Then the solutions $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ of (1.36) converge strongly in $L^2(0, T_0; H^{s'}_{loc}(\mathbb{R}))$ for all $s' < s$ to $(0, \overline{u}, \overline{\theta})$, where $(\overline{u}, \overline{\theta})$ is the unique solution of (1.37) with the initial data (w_{in}, θ_{in}), where w_{in} is determined by

$$w_{in} = \frac{1}{2}\kappa e^{\theta_{in}} \partial_x \theta_{in}. \tag{1.45}$$

Remark 1.10. In the proof of Theorem 1.9, we extend the convergence lemma of [1,35] to the different asymptotic states at infinity.

Remark 1.11. If $\theta^\varepsilon_{in} - \tilde{\theta} \to 0$ as $\varepsilon \to 0$, then one must have that $\bar{\theta} = \tilde{\theta}, \bar{u} = \frac{1}{2}\kappa e^{\tilde{\theta}} \partial_x \tilde{\theta}$ which is indeed the diffusive wave solution constructed in (1.15).

Remark 1.12. The condition (1.44) can also be replaced by $|\theta_{in}(x) - \theta_-| \leq C|x|^{-1-\sigma}$, for $x \in (-\infty, -1]$.

The paper is organized as follows. In Section 2 we consider the low Mach limit for the well-prepared data and in Section 3 for the ill-prepared data.

Notations: Throughout this paper, the positive generic constants independent of ε are denoted by c, C, C_i $(i \in \mathbb{N})$. And we will use $\|\cdot\|$ to denote the standard $L^2(\mathbb{R})$ norm, and $\|\cdot\|_{H^i}$ $(i \in \mathbb{N})$ to denote the Sobolev $H^i(\mathbb{R})$ norm. Sometimes, we also use $O(1)$ to denote a uniform bounded constant independent of ε.

2. Well-prepared data

This section is devoted to the low Mach limit for the well-prepared data. For simplicity, we omit the superscript ε of the variables throughout this section. We first reformulate the system (1.17) as follows.

Reformulation of the system (1.17):

Set the scaling

$$y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}. \tag{2.1}$$

In the following, we will also use the notations $(v, u, \mathcal{T})(\tau, y)$ and $(\tilde{v}, \tilde{u}, \tilde{\mathcal{T}})(\tau, y)$, etc., in the scaled independent variables. Set the perturbation around the profile $(\tilde{v}, \tilde{u}, \tilde{\mathcal{T}})(\tau, y)$ by

$$(\phi, \psi, \omega, \zeta)(\tau, y) = \Big(v - \tilde{v}, \varepsilon u - \varepsilon\tilde{u}, \mathcal{T} + \frac{1}{2}|\varepsilon u|^2 - \tilde{\mathcal{T}} - \frac{1}{2}|\varepsilon\tilde{u}|^2, \mathcal{T} - \tilde{\mathcal{T}}\Big)(\tau, y), \tag{2.2}$$

and introduce

$$(\Phi, \Psi, \tilde{W})(\tau, y) = \int_{-\infty}^{y} (\phi, \psi, \omega)(\tau, z)dz. \tag{2.3}$$

From (1.27), we note that $(\Phi, \Psi, \tilde{W})(0, \pm\infty) = 0$. And it follows from the equations of $(\Phi, \Psi, \tilde{W})$ in (2.4) below that $(\Phi, \Psi, \tilde{W})(\tau, \pm\infty) \equiv 0$ for $\tau > 0$.

Subtracting (1.23) from (1.17) and integrating the resulting system yield the following integrated system for $(\Phi, \Psi, \tilde{W})$:

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau + P - \tilde{P} = \tilde{\mu}\big(\frac{1}{v}\varepsilon u_y - \frac{1}{\tilde{v}}\varepsilon\tilde{u}_y\big) - \tilde{R}_1, \\ \tilde{W}_\tau + \varepsilon P u - \varepsilon\tilde{P}\tilde{u} = \kappa\big(\frac{1}{v}\mathcal{T}_y - \frac{1}{\tilde{v}}\tilde{\mathcal{T}}_y\big) + \tilde{\mu}\varepsilon^2\big(\frac{1}{v}uu_y - \frac{1}{\tilde{v}}\tilde{u}\tilde{u}_y\big) - \varepsilon\tilde{R}_2. \end{cases} \tag{2.4}$$

We note that the term involving time derivative in $(2.4)_3$ is $\tilde{W}_\tau$, while the dissipation term of $(2.4)_3$ is in the form $\kappa\big(\frac{1}{v}\mathcal{T}_y - \frac{1}{\tilde{v}}\tilde{\mathcal{T}}_y\big)$ which is closely related to the perturbation of temperature. To write the dissipation equation (2.4)$_3$ in a unified way and make the linearized formulation more convenient to do the energy estimates, instead of the variable $\tilde{W}$ which is the anti-derivative of the total energy, it is more convenient to introduce another variable W related to the temperature, *i.e.*,

$$W = \tilde{W} - \varepsilon\tilde{u}\Psi, \tag{2.5}$$

which yields that

$$\zeta = W_y - Y, \quad \text{with} \;\; Y = \frac{1}{2}|\Psi_y|^2 - \varepsilon\tilde{u}_y\Psi. \tag{2.6}$$

In terms of the new variable W, one linearize the system (2.4) as

$$\begin{cases} \Phi_\tau - \Psi_y = 0, \\ \Psi_\tau - \frac{1}{\tilde{v}}\Phi_y + \frac{1}{\tilde{v}}W_y = \tilde{\mu}\frac{1}{\tilde{v}}\Psi_{yy} + Q_1, \\ W_\tau + \Psi_y = \frac{\kappa}{\tilde{v}}W_{yy} + Q_2, \end{cases} \tag{2.7}$$

where

$$Q_1 = \tilde{\mu}(\frac{1}{v} - \frac{1}{\tilde{v}})\varepsilon u_y + J_1 + \frac{1}{\tilde{v}}Y - \tilde{R}_1, \tag{2.8}$$

$$Q_2 = \kappa(\frac{1}{v} - \frac{1}{\tilde{v}})\mathcal{T}_y + \tilde{\mu}\frac{\varepsilon u_y}{v}\Psi_y + J_2 - \varepsilon\tilde{u}_\tau\Psi - \frac{\kappa}{\tilde{v}}Y_y - \varepsilon\tilde{R}_2 + \varepsilon\tilde{u}\tilde{R}_1, \tag{2.9}$$

and

$$\begin{cases} J_1 = \frac{\tilde{P}-1}{\tilde{v}}\Phi_y - [P - \tilde{P} + \frac{\tilde{P}}{\tilde{v}}(v-\tilde{v}) - \frac{1}{\tilde{v}}(\mathcal{T} - \tilde{\mathcal{T}})] \\ \quad = O(1)(|\Phi_y|^2 + |W_y|^2 + Y^2 + |\varepsilon\tilde{u}|^4), \\ J_2 = (1-P)\Psi_y = O(1)(|(\Phi_y, \Psi_y, W_y)|^2 + Y^2 + |\varepsilon\tilde{u}|^4). \end{cases}$$

To prove Theorem 1.1, one needs only to show the following *a priori* estimate in the scaled independent variables and employ the continuity argument since the local existence for compressible Navier–Stokes system is known.

Proposition 2.1 *(A priori estimates). Assume that* (Φ, Ψ, W) *is a smooth solution of* (2.7) *with zero initial data in the time interval* $\tau \in [0, T]$*. There exist constants* $\delta_1 > 0$ *and* $\varepsilon_1 > 0$ *such that if* $\delta \le \delta_1$ *and* $\varepsilon \le \varepsilon_1$*, the following estimates hold*

$$\begin{cases} \|(\Phi, \Psi, W)(\tau)\|^2_{L^\infty_y} \le C\sqrt{\delta}\varepsilon, \\ \|(\phi, \psi, \zeta)(\tau)\|^2_{L^2_y} \le C\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}, \\ \|(\phi_y, \psi_y, \zeta_y)(\tau)\|^2_{L^2_y} \le C\sqrt{\delta}\varepsilon^3(1+\varepsilon^2\tau)^{-\frac{3}{2}+C_0\sqrt{\delta}}. \end{cases}$$

To obtain the above *a priori estimates*, one assumes the following *a priori* assumption:

$$N(T) = \sup_{0\le\tau\le T}\left\{\frac{1}{\varepsilon}\|(\Phi,\Psi,W)(\tau)\|_{L^\infty}^2 + \frac{1}{\varepsilon^2}\|(\phi,\psi,\zeta)(\tau)\|_{L^2}^2 + \frac{1}{\varepsilon^3}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|_{L^2}^2\right\} \le \chi^2, \tag{2.10}$$

where χ is a small constant determined later. It is noted that the local existence theorem yields that (2.10) holds for some positive time which may depend on $\varepsilon > 0$ and $\delta > 0$. We also point out that once we have proved the above Proposition 2.1, then it implies that (2.10) is valid, and hence we justify the use of above *a priori* assumption (2.10).

Basic estimates:

We start with the elementary energy estimates. Multiplying $(2.7)_1$ by Φ, $(2.7)_2$ by $\tilde{v}\Psi$, $(2.7)_3$ by W respectively, and adding all the resulting equations, one can obtain that

$$\begin{aligned}&\left(\frac{1}{2}\Phi^2 + \frac{1}{2}\tilde{v}\Psi^2 + \frac{1}{2}W^2\right)_\tau + \tilde{\mu}\Psi_y^2 + \frac{\kappa}{\tilde{v}}W_y^2\\ &= \frac{1}{2}\tilde{v}_\tau\Psi^2 + \tilde{v}Q_1\Psi + Q_2W - \left(\frac{\kappa}{\tilde{v}}\right)_y W_yW + (\cdots)_y.\end{aligned} \tag{2.11}$$

Define

$$m = (\Phi,\Psi,W)^t,$$

where $(\cdot,\cdot,\cdot)^t$ is the transpose of the vector $(\cdot,\cdot,\cdot)$, then from (2.7), one has

$$m_\tau + A_1m_y = A_2m_{yy} + A_3, \tag{2.12}$$

where

$$A_1 = \begin{pmatrix} 0 & -1 & 0\\ -\frac{1}{\tilde{v}} & 0 & \frac{1}{\tilde{v}}\\ 0 & 1 & 0\end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{\tilde{\mu}}{\tilde{v}} & 0\\ 0 & 0 & \frac{\kappa}{\tilde{v}}\end{pmatrix},\quad A_3 = (0,Q_1,Q_2)^t.$$

It follows from a direct computation that the eigenvalues of the matrix A_1 are $\lambda_1, 0, \lambda_3$ with $\lambda_3 = -\lambda_1 = \sqrt{\frac{2}{\tilde{v}}}$. The corresponding normalized left and right eigenvectors can be chosen as

$$\begin{aligned} l_1 &= \frac{1}{2}(-1,-\frac{2}{\lambda_3},1),\ l_2 = \sqrt{\frac{1}{2}}(1,0,1),\ l_3 = \frac{1}{2}(-1,\frac{2}{\lambda_3},1),\\ r_1 &= \frac{1}{2}(-1,-\lambda_3,1)^t,\ r_2 = \sqrt{\frac{1}{2}}(1,0,1)^t,\ r_3 = \frac{1}{2}(-1,\lambda_3,1)^t\end{aligned}$$

so that

$$l_i r_j = \delta_{ij},\ i,j = 1,2,3, \quad LA_1R = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where

$$L = (l_1, l_2, l_3)^t,\ R = (r_1, r_2, r_3).$$

Let

$$B = Lm = (b_1, b_2, b_3), \tag{2.13}$$

then multiplying the equations (2.12) by the matrix L yields that

$$B_\tau + \Lambda B_y = LA_2RB_{yy} + 2LA_2R_yB_y + [(L_\tau + \Lambda L_y)R + LA_2R_{yy}]B + LA_3. \tag{2.14}$$

We shall use a weighted energy method to derive the intrinsic dissipation. Without loss of generality, we assume that $\hat{\mathcal{T}}_y > 0$. The case that $\hat{\mathcal{T}}_y < 0$ can be treated in the same way. Let $\mathcal{T}_1 = \frac{\hat{\mathcal{T}}}{\mathcal{T}_+}$, then $|\mathcal{T}_1 - 1| \le C\delta$. Multiplying (2.14) by $\tilde{B} = (\mathcal{T}_1^N b_1, b_2, \mathcal{T}_1^{-N} b_3)$ with a large positive integer N which will be chosen later, one can get that

$$\begin{aligned}
&\left(\frac{\mathcal{T}_1^N}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{\mathcal{T}_1^{-N}}{2}b_3^2\right)_\tau - \left(\frac{\mathcal{T}_1^N}{2}\right)_\tau b_1^2 - \left(\frac{\mathcal{T}_1^{-N}}{2}\right)_\tau b_3^2 + \tilde{B}_yA_4B_y + \tilde{B}A_{4y}B_y \\
&-\frac{\mathcal{T}_1^{N-1}}{2}(N\lambda_1\mathcal{T}_{1y} + \mathcal{T}_1\lambda_{1y})b_1^2 + \frac{\mathcal{T}_1^{-N-1}}{2}(N\lambda_3\mathcal{T}_{1y} - \mathcal{T}_1\lambda_{3y})b_3^2 + (\cdots)_y \\
&= 2\tilde{B}LA_2R_yB_y + \tilde{B}[L_\tau R + LA_2R_{yy}]B + \tilde{B}\Lambda L_yRB + \tilde{B}LA_3.
\end{aligned} \tag{2.15}$$

A direct computation shows that the symmetric matrix $LA_2R = A_4$ is nonnegative. Define

$$E_1 = \int \left(\frac{1}{2}\Phi^2 + \frac{1}{2}\tilde{v}\Psi^2 + \frac{1}{2}W^2\right)dy + \int \left(\frac{\mathcal{T}_1^N}{2}b_1^2 + \frac{1}{2}b_2^2 + \frac{\mathcal{T}_1^{-N}}{2}b_3^2\right)dy, \tag{2.16}$$

$$K_1 = \int \left(\tilde{\mu}\Psi_y^2 + \frac{\kappa}{\tilde{v}}W_y^2 + B_yA_4B_y\right)dy. \tag{2.17}$$

Note that

$$\begin{aligned}
\left|\int (\tilde{B} - B)_y A_4 B_y dy\right| &\le C\delta \int |B_y|^2 dy + C\delta \int |\hat{\mathcal{T}}_y|^2 |B|^2 dy \\
&\le C\varepsilon^2\delta(1+t)^{-1}E_1 + C\delta K_1 + C\delta \int |\Phi_y|^2 dy.
\end{aligned} \tag{2.18}$$

Similarly, the terms $\tilde{B}A_{4y}B_y$, $\tilde{B}LA_2R_yB_y$ and $\tilde{B}[L_\tau R + LA_2R_{yy}]B$ in (2.15) satisfy the same estimate. For $\tilde{B}\Lambda L_yRB$ and $\tilde{B}LA_3$, we need to use the explicit presentation. By the choice of the characteristic matrix L and R, one has that

$$\Lambda L_y R = \frac{1}{2}\lambda_{3y}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad LA_3 = \left(\frac{1}{2}Q_2 - \frac{Q_1}{\lambda_3},\ \sqrt{\frac{1}{2}}Q_2,\ \frac{1}{2}Q_2 + \frac{Q_1}{\lambda_3}\right)^t,$$

which yields immediately that

$$|\tilde{B}\Lambda L_y RB| \le C|\lambda_{3y}|(b_1^2 + b_3^2), \quad \tilde{B}LA_3 \le C|(b_1, b_3)| \cdot |(Q_1, Q_2)| + C|b_2 Q_2|. \quad (2.19)$$

Integrating (2.11) over $\mathbb{R}$ with respect to y, using (2.15)–(2.18), (2.19) and the Cauchy inequality, and choosing N sufficiently large, we obtain that

$$E_{1\tau} + \frac{1}{2}K_1 + 2\int |\hat{\mathcal{T}}_y|(b_1^2 + b_3^2)|dy \le C\varepsilon^2\delta(1+t)^{-1}(E_1+1) + C\delta K_1 + C\delta\int \Phi_y^2 dy + I, \quad (2.20)$$

with

$$I = C\int |(b_1, b_3)| \cdot |(Q_1, Q_2)| + |b_2 Q_2| dy. \quad (2.21)$$

Here, in the proof of (2.20), we have used the fact

$$\Psi = \frac{1}{2}\lambda_3(b_3 - b_1), \quad (2.22)$$

and for N large enough that

$$-\frac{1}{2}\mathcal{T}_1^{N-1}(N\lambda_1\hat{\mathcal{T}}_{1y} + 2\mathcal{T}_1\lambda_{1y})b_1^2 + \frac{1}{2}\mathcal{T}_1^{-N-1}(N\lambda_3\mathcal{T}_{1y} - 2\mathcal{T}_1\lambda_{3y})b_3^2 - \tilde{B}\Lambda L_y RB$$
$$\ge 3|\hat{\mathcal{T}}_y|(b_1^2 + b_3^2). \quad (2.23)$$

Although Q_1 contains the term $\tilde{R}_1$ with the decay rate $\varepsilon^2(1+t)^{-1}$, the terms in (2.21) involving Q_1 can be estimated by the intrinsic dissipation on b_1 and b_3 as shown later. The terms involving Q_2 contains the term $\varepsilon\tilde{R}_2$ which has a better decay rate $\varepsilon^3(1+t)^{-\frac{3}{2}}$ and can be estimated directly. For brevity, we only estimate $\int |Q_1 b_1| dy$ and $\int |Q_2 b_2| dy$ as follows for illustration.

Estimation on $\int |Q_1 b_1| dy$:

It follows from (2.8) that

$$\int |Q_1 b_1| dy \le \int \left|\left[\left(\frac{\tilde{\mu}}{v} - \frac{\tilde{\mu}}{\tilde{v}}\right)u_y + J_1 + \frac{Y}{\tilde{v}}\right] \cdot b_1\right| + |\tilde{R}_1 b_1| dy =: I_1 + I_2. \quad (2.24)$$

A direct calculation shows that

$$\int \left|\left(\frac{\tilde{\mu}}{v} - \frac{\tilde{\mu}}{\tilde{v}}\right)\varepsilon u_y \cdot b_1\right| dy \le C(\chi+\delta)(K_1 + \|\Phi_y\|_{L^2}^2 + \|\psi_y\|_{L^2}^2) + C\delta\varepsilon^4(1+t)^{-2}\|\Phi\|_{L^2}^2$$
$$+ C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^5(1+t)^{-\frac{5}{2}},$$

and

$$\int \Big(|J_1| + |\frac{Y}{\tilde{v}}|\Big) \cdot |b_1| dy \leq C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2 + \|\psi_y\|_{L^2}^2) + C\delta\varepsilon^4(1+t)^{-2}\|\Phi\|_{L^2}^2 \\ + C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^5(1+t)^{-\frac{5}{2}},$$

which yields immediately that

$$I_1 \leq C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2 + \|\psi_y\|_{L^2}^2) + +C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^5(1+t)^{-\frac{5}{2}}. \quad (2.25)$$

On the other hand, it follows from (1.24) and the Cauchy inequality that

$$\int |\tilde{R}_1 b_1| dy \leq \delta \int |\hat{\mathcal{T}}_y| b_1^2 dy + C\delta\varepsilon^2(1+t)^{-1},$$

which, together with (2.25), yields that

$$\int |Q_1 b_1| dy \leq \delta \int |\hat{\mathcal{T}}_y| b_1^2 dy + C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2 + \|\psi_y\|_{L^2}^2) \\ + C\delta\varepsilon^2(1+t)^{-1}(E_1+1). \quad (2.26)$$

Estimation on $\int |Q_2 b_2| dy$:

It follows from (2.9) that

$$\int |Q_2 b_2| dy \leq \int \Big|\Big(\frac{\kappa}{v} - \frac{\kappa}{\tilde{v}}\Big)\mathcal{T}_y + \frac{\tilde{\mu}}{v}\varepsilon u_y \Psi_y + J_2 - \varepsilon\tilde{u}_\tau\Psi - \frac{\kappa}{\tilde{v}}Y_y - \varepsilon\tilde{R}_2 + \varepsilon\tilde{u}\tilde{R}_1\Big| \cdot |b_2| dy. \quad (2.27)$$

The Cauchy inequality yields that

$$\int \Big|\Big(\frac{\kappa}{v} - \frac{\kappa}{\tilde{v}}\Big)\mathcal{T}_y + \frac{\tilde{\mu}}{v}\varepsilon u_y \Psi_y\Big| \cdot |b_2| dy \\ \leq C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2) + C\chi\|(\psi_y, \zeta_y)\|_{L^2}^2 + C\delta\varepsilon^2(1+t)^{-1}E_1, \quad (2.28)$$

and

$$\int \Big|J_2 - \varepsilon\tilde{u}_\tau\Psi - \frac{\kappa}{\tilde{v}}Y_y - \varepsilon\tilde{R}_2 + \varepsilon\tilde{u}\tilde{R}_1\Big| \cdot |b_2| dy \\ \leq C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2) + C\chi\|\psi_y\|_{L^2}^2 + C\delta\varepsilon^2(1+t)^{-1}(E_1+1). \quad (2.29)$$

Thus combining (2.28), (2.29) and using the Cauchy inequality, one has that

$$\int |Q_2 b_2| dy \leq C(\delta+\chi)(K_1 + \|\Phi_y\|_{L^2}^2) + C\chi\|(\psi_y, \zeta_y)\|_{L^2}^2 \\ + C\delta\varepsilon^2(1+t)^{-1}(E_1+1). \quad (2.30)$$

Substituting (2.26), (2.30) into (2.20), one obtains that

$$
\begin{aligned}
&E_{1\tau} + \frac{1}{4}K_1 + \int |\hat{\mathcal{T}}_y|(b_1^2 + b_3^2)|dy \\
&\le C\delta\varepsilon^2(1+t)^{-1}(E_1+1) + C(\delta+\chi)(\|\Phi_y\|_{L^2}^2 + \|(\psi_y,\zeta_y)\|_{L^2}^2).
\end{aligned}
\tag{2.31}
$$

Note that the norm $\|\Phi_y\|_{L^2}^2$ is not included in K_1. To complete the lower-order inequality, we need to estimate Φ_y. Using $(2.7)_1$, we rewrite $(2.7)_2$ to be the following form

$$
\frac{\tilde{\mu}}{\tilde{v}}\Phi_{y\tau} - \Psi_\tau + \frac{1}{\tilde{v}}\Phi_y = \frac{1}{\tilde{v}}W_y - Q_1. \tag{2.32}
$$

Multiplying (2.32) by Φ_y yields that

$$
\Big(\frac{\tilde{\mu}}{2\tilde{v}}\Phi_y^2\Big)_\tau - \Big(\frac{\tilde{\mu}}{2\tilde{v}}\Big)_\tau\Phi_y^2 - \Phi_y\Psi_\tau + \frac{1}{\tilde{v}}\Phi_y^2 = \Big(\frac{1}{\tilde{v}}W_y - Q_1\Big)\Phi_y. \tag{2.33}
$$

Noting

$$
\Psi_\tau\Phi_y = (\Psi\Phi_y)_\tau - (\Psi\Phi_\tau)_y + \Psi_y^2,
$$

which, together with (2.33) and integrating by parts, yields that

$$
\Big(\int \frac{\tilde{\mu}}{2\tilde{v}}\Phi_y^2 - \Phi_y\Psi dy\Big)_\tau + \int \frac{1}{2\tilde{v}}\Phi_y^2 dy \le C\int \Psi_y^2 + W_y^2 dy + C\delta(1+t)^{-3/2} + \int Q_1^2 dy. \tag{2.34}
$$

On the other hand, it follows from (2.8) and (2.10) that

$$
\int Q_1^2 dy \le C(\delta+\chi)(K_1 + \|\Phi_y\|^2 + \|\psi_y\|^2) + C\delta\varepsilon^2(1+t)^{-1}E_1 + C\delta\varepsilon^2(1+t)^{-1}. \tag{2.35}
$$

Substituting (2.35) into (2.34), one has that

$$
\begin{aligned}
&\Big(\int \frac{\tilde{\mu}}{2\tilde{v}}\Phi_y^2 - \Phi_y\Psi dy\Big)_\tau + \int \frac{1}{4\tilde{v}}\Phi_y^2 dy \\
&\le C_1K_1 + C_1\delta\varepsilon^2(1+t)^{-1}(E_1+1) + C_1(\delta+\chi)\|\psi_y\|^2.
\end{aligned}
\tag{2.36}
$$

By choosing $\tilde{C}_1$ large enough, it holds that

$$
\tilde{C}_1E_1 + \int \frac{\tilde{\mu}}{2\tilde{v}}\Phi_y^2 - \Phi_y\Psi dy \ge \frac{1}{2}\tilde{C}_1E_1 + \int \frac{\tilde{\mu}}{4\tilde{v}}\Phi_y^2 dy, \quad \text{and} \quad \frac{1}{4}\tilde{C}_1 - C_1 \ge \frac{1}{8}\tilde{C}_1. \tag{2.37}
$$

Thus, multiplying (2.31) by $\tilde{C}_1$ and using (2.37), one obtains the low order estimates:

Lemma 2.2. *If δ and χ are suitably small, then it holds that*

$$E_{2\tau} + K_2 + \int |\hat{\mathcal{T}}_y|(b_1^2 + b_3^2)|dy \le C\delta\varepsilon^2(1+t)^{-1}(E_1+1) + C(\delta+\chi)\|(\psi_y,\zeta_y)\|^2,$$

where

$$E_2 = \tilde{C}_1 E_1 + \int \frac{\tilde{\mu}}{2\tilde{v}}\Phi_y^2 - \Phi_y\Psi dy, \quad K_2 = \frac{1}{8}\tilde{C}_1 K_1 + \int \frac{1}{8\tilde{v}}\Phi_y^2 dy.$$

Derivative estimates:

To obtain the estimate for the first-order derivative of $\|(\Phi_y, \Psi_y, W_y)\|$, we shall use an energy estimate based on the convex entropy of the compressible Navier–Stokes equations. Applying ∂_y to (2.4) yields that

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + (P - \tilde{P})_y = \varepsilon(\frac{\tilde{\mu}}{v}u_y - \frac{\tilde{\mu}}{\tilde{v}}\tilde{u}_y)_y - \tilde{R}_{1y}, \\ \zeta_\tau + \varepsilon(Pu_y - \tilde{P}\tilde{u}_y) = (\frac{\kappa}{v}\mathcal{T}_y - \frac{\kappa}{\tilde{v}}\tilde{\mathcal{T}}_y)_y + Q_3, \end{cases} \tag{2.38}$$

where

$$Q_3 = \frac{\tilde{\mu}}{v}\varepsilon^2 u_y^2 - \varepsilon^2\Big(\frac{\tilde{\mu}}{\tilde{v}}\tilde{u}\tilde{u}_y\Big)_y + \varepsilon\tilde{P}_y\tilde{u} + \frac{1}{2}(\varepsilon^2\tilde{u}^2)_\tau - \varepsilon\tilde{R}_{2y}.$$

Lemma 2.3. *If δ and χ are suitably small, then it holds that*

$$E_{3\tau} + \frac{1}{2}K_3 \le C\delta\varepsilon^2(1+t)^{-1}E_3 + \varepsilon^2(1+t)^{-1}\int |\hat{\mathcal{T}}_y|(b_1^2+b_3^2)dy + C\delta\varepsilon^4(1+t)^{-2}, \tag{2.39}$$

where

$$E_3 = \int \frac{1}{2}\psi^2 + \tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big) + \tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{\mathcal{T}}{\tilde{\mathcal{T}}}\Big)dy, \quad K_3 = \int \frac{\tilde{\mu}}{v}\psi_y^2 + \frac{3}{4}\frac{\kappa}{v\mathcal{T}}\zeta_y^2 dy.$$

Proof. Multiplying (2.38)$_2$ by ψ yields that

$$\Big(\frac{1}{2}\psi^2\Big)_\tau - (P-\tilde{P})\psi_y + \varepsilon\Big(\frac{\tilde{\mu}}{v}u_y - \frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_y\Big)\psi_y = -\tilde{R}_{1y}\psi + (\cdots)_y.$$

Since $P - \tilde{P} = \tilde{\mathcal{T}}(\frac{1}{v} - \frac{1}{\tilde{v}}) + \frac{\zeta}{v}$, as in [17], it holds that

$$\Big(\frac{1}{2}\psi^2\Big)_\tau - \tilde{\mathcal{T}}\Big(\frac{1}{v} - \frac{1}{\tilde{v}}\Big)\phi_\tau - \frac{\zeta}{v}\psi_y + \frac{\tilde{\mu}}{v}\psi_y^2 + \Big(\frac{\tilde{\mu}}{v} - \frac{\tilde{\mu}}{\tilde{v}}\Big)\varepsilon\tilde{u}_y\psi_y = -\tilde{R}_{1y}\psi + (\cdots)_y. \tag{2.40}$$

Denote

$$\hat{\Phi}(s) = s - 1 - \ln s, \tag{2.41}$$

then, it is straightforward to check that $\hat{\Phi}'(1) = 0$ and $\hat{\Phi}(s)$ is strictly convex around $s = 1$. Moreover, it holds that

$$\Big(\tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big)\Big)_\tau = \tilde{\mathcal{T}}_\tau\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big) + \tilde{\mathcal{T}}\Big(-\frac{1}{v}+\frac{1}{\tilde{v}}\Big)\phi_\tau + \tilde{\mathcal{T}}\Big(-\frac{v}{\tilde{v}^2}+\frac{1}{\tilde{v}}\Big)\tilde{v}_\tau + \tilde{\mathcal{T}}\Big(-\frac{1}{v}+\frac{1}{\tilde{v}}\Big)\tilde{v}_\tau$$
$$= \tilde{\mathcal{T}}\Big(-\frac{1}{v}+\frac{1}{\tilde{v}}\Big)\phi_\tau - \tilde{P}\hat{\Psi}\Big(\frac{v}{\tilde{v}}\Big)\tilde{v}_\tau + \tilde{v}\tilde{P}_\tau\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big), \tag{2.42}$$

with $\hat{\Psi}(s) = \hat{\Phi}(s^{-1})$. Therefore, it follows from (2.40) and (2.42) that

$$\Big(\frac{1}{2}\psi^2 + \tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big)\Big)_\tau + \tilde{P}\tilde{v}_\tau\hat{\Psi}\Big(\frac{v}{\tilde{v}}\Big) - \frac{\zeta}{v}\psi_y + \frac{\tilde{\mu}}{v}\psi_y^2 + \Big(\frac{\tilde{\mu}}{v}-\frac{\tilde{\mu}}{\tilde{v}}\Big)\varepsilon\tilde{u}_y\psi_y$$
$$= \tilde{v}\tilde{P}_\tau\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big) - \tilde{R}_{1y}\psi + (\cdots)_y. \tag{2.43}$$

On the other hand, multiplying $(2.38)_3$ by $\frac{\zeta}{\mathcal{T}}$ yields that

$$\frac{\zeta}{\mathcal{T}}\zeta_\tau + \varepsilon(Pu_y - \tilde{P}\tilde{u}_y)\frac{\zeta}{\mathcal{T}} = \Big(\frac{\kappa}{v}\mathcal{T}_y - \frac{\kappa}{\tilde{v}}\tilde{\mathcal{T}}_y\Big)_y\frac{\zeta}{\mathcal{T}} + Q_3\frac{\zeta}{\mathcal{T}}. \tag{2.44}$$

A direct calculation gives that

$$\frac{\zeta}{\mathcal{T}}\zeta_\tau = \Big(\tilde{\mathcal{T}}\hat{\Phi}(\frac{\mathcal{T}}{\tilde{\mathcal{T}}})\Big)_\tau + O(1)|\tilde{\mathcal{T}}_\tau|\zeta^2, \tag{2.45}$$

$$\varepsilon(Pu_y - \tilde{P}\tilde{u}_y)\frac{\zeta}{\mathcal{T}} = \frac{\zeta}{v}\psi_y + (P-\tilde{P})\varepsilon\tilde{u}_y\frac{\zeta}{\mathcal{T}}, \tag{2.46}$$

and

$$\Big(\frac{\kappa}{v}\mathcal{T}_y - \frac{\kappa}{\tilde{v}}\tilde{\mathcal{T}}_y\Big)_y\frac{\zeta}{\mathcal{T}} \le -\frac{3}{4}\frac{\kappa}{v\mathcal{T}}\zeta_y^2 + O(1)|\tilde{\mathcal{T}}_y|^2|(\phi,\zeta)|^2 + (\cdots)_y. \tag{2.47}$$

Substitute (2.45)–(2.47) into (2.44), one obtains that

$$\Big(\tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{\mathcal{T}}{\tilde{\mathcal{T}}}\Big)\Big)_\tau + \frac{\zeta}{v}\psi_y + \frac{3}{4}\frac{\kappa}{v\mathcal{T}}\zeta_y^2 \le C\delta\varepsilon^2(1+t)^{-1}|(\phi,\zeta)|^2 + Q_3\frac{\zeta}{\mathcal{T}} + (\cdots)_y. \tag{2.48}$$

It follows from (2.48) and (2.43) that

$$\Big(\frac{1}{2}\psi^2 + \tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{v}{\tilde{v}}\Big) + \tilde{\mathcal{T}}\hat{\Phi}\Big(\frac{\mathcal{T}}{\tilde{\mathcal{T}}}\Big)\Big)_\tau + \frac{\tilde{\mu}}{v}\psi_y^2 + \frac{3}{4}\frac{\kappa}{v\mathcal{T}}\zeta_y^2$$
$$\le C\delta\varepsilon^2(1+t)^{-1}|(\phi,\zeta)|^2 - \tilde{R}_{1y}\psi + Q_3\frac{\zeta}{\mathcal{T}} + (\cdots)_y. \tag{2.49}$$

Using (2.22), one has that

$$\Big|\int \tilde{R}_{1y}\psi dy\Big| = \Big|\int \tilde{R}_{1yy}\Psi dy\Big| \le \varepsilon^2(1+t)^{-1}\int |\hat{\mathcal{T}}_y|(b_1^2+b_3^2)dy + C\delta\varepsilon^4(1+t)^{-2}, \tag{2.50}$$

and

$$\begin{aligned}\Big|\int Q_3\frac{\zeta}{\theta}dy\Big| &\le C\int|\zeta|\cdot|\psi_y|^2dy+C\delta\varepsilon^4\int|\zeta|(1+t)^{-2}e^{-\frac{cx^2}{1+t}}dy\\ &\le C\chi\|\psi_y\|^2+C\delta\varepsilon^2(1+t)^{-1}\|\zeta\|^2+C\delta\varepsilon^5(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{2.51}$$

Integrating (2.49) with respect to y, using (2.50) and (2.51), then one obtains (2.39). Thus, the proof of Lemma 2.3 is completed. □

Since the norm $\|\phi_y\|^2$ is not included in K_3. To complete the first derivative inequality, we follow the same way for Φ_y. We rewrite the equation $(2.38)_2$ as

$$\frac{\tilde\mu}{\tilde v}\phi_{y\tau}-\psi_\tau+(P-\tilde P)_y=-\Big(\frac{\tilde\mu}{\tilde v}\Big)_y\phi_y-\Big(\Big(\frac{\tilde\mu}{v}-\frac{\tilde\mu}{\tilde v}\Big)\varepsilon u_y\Big)_y+\tilde R_{1y}.\tag{2.52}$$

Lemma 2.4. *If δ and χ are suitably small, then it holds that*

$$\begin{aligned}&\Big(\int\frac{\tilde\mu}{2\tilde v}\phi_y^2-\phi_y\psi dy\Big)_\tau+\int\frac{\tilde P}{2\tilde v}\phi_y^2dy\\ &\le C_2K_3+C_2\delta\varepsilon^2(1+t)^{-1}E_3+C_2\delta\varepsilon^5(1+t)^{-\frac{5}{2}}+C_2(\|(\phi,\psi)\|_{L^\infty}^2+\|\psi_y\|^2)\|\psi_{yy}\|^2.\end{aligned}\tag{2.53}$$

Proof. Multiplying (2.52) by ϕ_y yields that

$$\Big(\frac{\tilde\mu}{2\tilde v}\phi_y^2\Big)_\tau-\psi_\tau\phi_y-(P-\tilde P)_y\phi_y=\Big(\frac{\tilde\mu}{2\tilde v}\Big)_\tau\phi_y^2+\Big\{-(\frac{\tilde\mu}{\tilde v})_y\psi_y-[(\frac{\tilde\mu}{v}-\frac{\tilde\mu}{\tilde v})\varepsilon u_y]_y\Big\}\phi_y+\tilde R_{1y}\phi_y.$$

A direct calculation yields that

$$-(P-\tilde P)_y=\frac{\tilde P}{\tilde v}\phi_y-\frac{1}{\tilde v}\zeta_y+\Big(\frac{P}{v}-\frac{\tilde P}{\tilde v}\Big)v_y-\Big(\frac{1}{v}-\frac{1}{\tilde v}\Big)\mathcal{T}_y,$$

and

$$\phi_y\psi_\tau=(\phi_y\psi)_\tau-(\phi_\tau\psi)_y+\psi_y^2.$$

Then combining the above three equations, one obtains that

$$\begin{aligned}&\Big(\int\frac{\tilde\mu}{2\tilde v}\phi_y^2-\phi_y\psi dy\Big)_\tau+\int\frac{\tilde P}{2\tilde v}\phi_y^2dy\\ &\le C_2K_3+C_2\delta\varepsilon^2(1+t)^{-1}E_3+C_2\delta\varepsilon^5(1+t)^{-\frac{5}{2}}+C_2(\|(\phi,\psi)\|_{L^\infty}^2+\|\psi_y\|^2)\|\psi_{yy}\|^2,\end{aligned}\tag{2.54}$$

where we have used

$$-\int(P-\tilde P)_y\phi_ydy\ge\int\frac{3\tilde P}{4\tilde v}\phi_y^2dy-C\|\zeta_y\|^2-C\delta\varepsilon^2(1+t)^{-1}E_3,$$

$$
\begin{aligned}
&\Big|\int (\frac{\tilde{\mu}}{\tilde{v}})_y \psi_y dy\Big| + \Big|\int [(\frac{\tilde{\mu}}{v} - \frac{\tilde{\mu}}{\tilde{v}})\varepsilon u_y]_y \phi_y dy\Big| + \Big|\int \tilde{R}_{1y}\phi_y dy\Big| \\
&\le \int \frac{\tilde{P}}{8\tilde{v}}\phi_y^2 dy + CK_3 + C\delta\varepsilon^2(1+t)^{-1}E_3 + C_2\delta\varepsilon^5(1+t)^{-\frac{5}{2}} \\
&\quad + \int |\phi_y|^2|\psi_y| + |\phi_y\psi_y\zeta_y| dy,
\end{aligned}
$$

and

$$
\begin{aligned}
\int |\phi_y|^2|\psi_y| + |\phi_y\psi_y\zeta_y|dy &\le C\|\phi_y\|^2\|\psi_y\|^{\frac{1}{2}}\|\psi_{yy}\|^{\frac{1}{2}} + C\|\phi_y\|\|\zeta_y\|\|\psi_y\|^{\frac{1}{2}}\|\psi_{yy}\|^{\frac{1}{2}} \\
&\le \int \frac{\tilde{P}}{16\tilde{v}}\phi_y^2 dy + C\|\zeta_y\|^2 + C\|\psi_y\|^2\|\psi_{yy}\|^2.
\end{aligned}
$$

Therefore the proof of Lemma 2.4 is completed. □

We now derive the higher order estimates. Applying ∂_y to (2.38) yields that

$$
\begin{cases}
\phi_{y\tau} - \psi_{yy} = 0, \\
\psi_{y\tau} + \frac{1}{\tilde{v}}\zeta_{yy} - \frac{1}{\tilde{v}}\phi_{yy} = \tilde{\mu}(\frac{1}{v}\varepsilon u_y - \frac{1}{\tilde{v}}\varepsilon\tilde{u}_y)_{yy} + Q_4 - \tilde{R}_{1yy}, \\
\zeta_{y\tau} + \psi_{yy} = (\frac{\kappa}{v}\mathcal{T}_y - \frac{\kappa}{\tilde{v}}\tilde{\mathcal{T}}_y)_{yy} + Q_5,
\end{cases}
\tag{2.55}
$$

where

$$
\begin{aligned}
Q_4 &= \frac{\tilde{P}-1}{\tilde{v}}\phi_{yy} + \Big(\frac{P}{v} - \frac{\tilde{P}}{\tilde{v}}\Big)\phi_{yy} - 2\Big(\frac{1}{v^2}\mathcal{T}_y v_y - \frac{1}{\tilde{v}^2}\tilde{\mathcal{T}}_y\tilde{v}_y\Big) + 2\Big(\frac{1}{v^3}v_y^2 - \frac{1}{\tilde{v}^3}\tilde{v}_y^2\Big) \\
&\quad + O(1)\Big(|\tilde{v}_{yy}||(\phi,\psi)| + |\phi\zeta_{yy}|\Big), \qquad (2.56) \\
Q_5 &= -\varepsilon\tilde{u}_{yy}(P - \tilde{P}) - \varepsilon(P_y u_y - \tilde{P}_y\tilde{u}_y) + (1-\tilde{P})\psi_{yy} + Q_{3y}. \qquad (2.57)
\end{aligned}
$$

Lemma 2.5. *If δ and χ are suitably small, then it holds that*

$$
\begin{aligned}
E_{4\tau} + \frac{1}{2}K_4 &\le C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3 + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}} \\
&\quad + C\|(\phi,\psi)\|\cdot\|(\phi_y,\psi_y,\zeta_y)\|^3 + C\delta\varepsilon^6(1+t)^{-3},
\end{aligned}
\tag{2.58}
$$

where

$$
E_4 = \int \frac{1}{2}\phi_y^2 + \frac{1}{2}\tilde{v}\psi_y^2 + \frac{1}{2}\zeta_y^2 dy, \quad K_4 = \int \frac{\tilde{\mu}\tilde{v}}{v}\psi_{yy}^2 + \frac{\kappa}{v}\zeta_{yy}^2 dy.
$$

Proof. Multiplying $(2.55)_1$ by ϕ_y, $(2.55)_2$ by $\tilde{v}\psi_y$, $(2.55)_3$ by ζ_y and adding the resulting equations, one obtains that

$$\begin{aligned}&\Big(\frac{1}{2}\phi_y^2+\frac{1}{2}\tilde{v}\psi_y^2+\frac{1}{2}\zeta_y^2\Big)_\tau+\frac{\tilde{\mu}\tilde{v}}{v}\psi_{yy}^2+\frac{\kappa}{v}\zeta_{yy}^2\\&=\frac{1}{2}\tilde{v}_\tau\psi_y^2+\tilde{v}Q_4\psi_y-\tilde{R}_{1yy}\tilde{v}\psi_y+Q_5\zeta_y+J_3+J_4+(\cdots)_y,\end{aligned}\tag{2.59}$$

where

$$\begin{aligned}J_3&=-\tilde{\mu}\Big(\frac{1}{v}\varepsilon u_y-\frac{1}{\tilde{v}}\varepsilon\tilde{u}_y\Big)_y\tilde{v}_y\psi_y-\Big(\frac{\tilde{\mu}}{v}\Big)_y\psi_{yy}\psi_y-\Big(\Big(\frac{\tilde{\mu}}{v}-\frac{\tilde{\mu}}{\tilde{v}}\Big)\varepsilon\tilde{u}_y\Big)_y\tilde{v}\psi_{yy},\\J_4&=-\Big(\frac{\kappa}{v}\Big)_y\zeta_{yy}\zeta_y-\Big(\Big(\frac{\kappa}{v}-\frac{\kappa}{\tilde{v}}\Big)\tilde{\theta}_y\Big)_y\zeta_{yy}.\end{aligned}$$

It is direct to obtain

$$\begin{aligned}&\Big|\int\Big\{\frac{\tilde{P}-1}{\tilde{v}}\phi_{yy}+\Big(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\Big)\phi_{yy}\Big\}\tilde{v}\psi_y dy\Big|\\&=\Big|\int\Big\{(\tilde{P}-1)\psi_y+\tilde{v}\psi_y\Big(\frac{P}{v}-\frac{\tilde{P}}{\tilde{v}}\Big)\Big\}_y\phi_y dy\Big|\\&\le(\delta+\chi)\|\psi_{yy}\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2+C\delta\varepsilon^4(1+t)^{-2}E_3\\&\quad+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}+C\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3,\end{aligned}\tag{2.60}$$

$$\begin{aligned}&2\Big|\int\Big(\frac{1}{v^2}\mathcal{T}_y v_y-\frac{1}{\tilde{v}^2}\tilde{\mathcal{T}}_y\tilde{v}_y\Big)\tilde{v}\psi_y dy\Big|+2\Big|\int\Big(\frac{1}{v^3}v_y^2-\frac{1}{\tilde{v}^3}\tilde{v}_y^2\Big)\tilde{v}\psi_y dy\Big|\\&\le\chi\|\psi_{yy}\|^2+C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3\\&\quad+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}},\end{aligned}\tag{2.61}$$

and

$$\begin{aligned}&\int\big(|\tilde{v}_{yy}||(\phi,\psi)|+|\phi\zeta_{yy}|\big)|\tilde{v}\psi_y|dy\\&\le\chi\|\zeta_{yy}\|^2+C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2\\&\quad+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3+C\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3.\end{aligned}\tag{2.62}$$

Then it follows from (2.56) and (2.60)–(2.62) that

$$\begin{aligned}\Big|\int\tilde{v}Q_4\psi_y dy\Big|&\le(\delta+\chi)\|(\psi_{yy},\zeta_{yy})\|^2+C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2+C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3\\&\quad+C\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3+C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}.\end{aligned}\tag{2.63}$$

Using Cauchy inequality, it is easy to get that

$$\Big|\int\tilde{v}\tilde{R}_{1yy}\psi_y dy\Big|\le C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|\psi_y\|^2+C\delta\varepsilon^6(1+t)^{-3}.\tag{2.64}$$

A direct calculation also yields that

$$\Big|\int Q_5\zeta_y dy\Big| \le (\delta+\chi)\|(\psi_{yy},\zeta_{yy})\|^2 + C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3 + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}, \tag{2.65}$$

and

$$\int |J_3|+|J_4|dy \le (\delta+\chi)\|(\psi_{yy},\zeta_{yy})\|^2 + C\delta\varepsilon(1+t)^{-\frac{1}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^2 + C\delta\varepsilon^3(1+t)^{-\frac{3}{2}}E_3 + C\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}}. \tag{2.66}$$

Integrating (2.59) with respect to y and using (2.63)–(2.66), one obtains (2.58). Therefore the proof of Lemma 2.5 is completed. □

We choose a large constants $\tilde{C}_2 > 1$ so that

$$\tilde{C}_2 E_3 + \int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_y^2 - \psi\phi_y dy \ge \frac{1}{2}\tilde{C}_2E_3 + \int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_y^2 dy, \quad \frac{1}{2}\tilde{C}_2 - C_2 \ge \frac{1}{4}\tilde{C}_2, \tag{2.67}$$

and define

$$E_5 = \tilde{C}_2E_3 + \int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_y^2 - \psi\phi_y dy + \varepsilon^{-1}E_4, \quad K_5 = \frac{1}{4}\tilde{C}_2K_3 + \int \frac{\tilde{p}}{4\tilde{v}}\phi_y^2 dy + \frac{1}{2}\varepsilon^{-1}K_4. \tag{2.68}$$

Thus, combining (2.39), (2.53) and (2.58), one obtains

Lemma 2.6. *If δ and χ are suitably small, then it holds that*

$$E_{5\tau} + K_5 \le C_3\varepsilon^2(1+t)^{-1}\int |\hat{\mathcal{T}}_y|(b_1^2+b_2^2)dy + C_3\delta\varepsilon^2(1+t)^{-\frac{3}{2}}E_5 + C_3\delta\varepsilon^4(1+t)^{-2}. \tag{2.69}$$

Proof of Proposition 2.1. Let $\tilde{C}_3 = 1 + C_3$, then from (2.36) and (2.69), one gets that

$$E_{6\tau} + K_6 \le C_0\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1}(E_6+\sqrt{\delta}),$$

where

$$E_6 = \tilde{C}_3E_2 + \varepsilon^{-2}E_5, \quad K_6 = \frac{1}{4}\tilde{C}_3K_2 + \frac{1}{2}\varepsilon^{-2}K_5.$$

Then the Gronwall's inequality yields that

$$E_6(\tau) \le C_0\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}, \quad \int_0^\tau K_6(s)ds \le CC_0\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \tag{2.70}$$

Since $c_3\|(\Phi,\Psi,W)\|^2 \le E_6$ for some positive constant c_3, one obtains that

$$\|(\Phi, \Psi, W)\|^2 \le CC_0\sqrt{\delta}(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \tag{2.71}$$

Multiplying (2.69) by $(1+\varepsilon^2\tau)$, one has that

$$\begin{aligned}[(1+\varepsilon^2\tau)E_5]_\tau + (1+\varepsilon^2\tau)K_5 &\le C\varepsilon^2 \int |\hat{\mathcal{T}}_y|(b_1^2+b_2^2)dy + C\varepsilon^2 E_5 + C_3\delta\varepsilon^4(1+\varepsilon^2\tau)^{-1} \\ &\le C\varepsilon^2 K_6 + C_3\delta\varepsilon^4(1+\varepsilon^2\tau)^{-1},\end{aligned} \tag{2.72}$$

where we have used the fact that

$$\begin{aligned}E_5(\tau) &\le C\|(\phi,\psi,\zeta)\|^2 + C\varepsilon^{-1}\|(\phi_y,\psi_y,\zeta_y)\|^2 \\ &\le C\|(\Phi_y,\Psi_y,W_y)\|^2 + C\varepsilon^{-1}\|(\phi_y,\psi_y,\zeta_y)\|^2 + C\delta\varepsilon^4(1+\varepsilon^2\tau)^{-\frac{3}{2}} \\ &\le CK_6 + C\delta\varepsilon^4(1+\varepsilon^2\tau)^{-\frac{3}{2}}.\end{aligned}$$

Integrating (2.72) over $[0,\tau]$ and using (2.70), one obtains that

$$E_5 \le CC_0\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}, \qquad \int_0^\tau (1+\varepsilon^2 s)K_5 ds \le CC_0\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}. \tag{2.73}$$

Furthermore, it holds that

$$\begin{aligned}E_5(\tau) &\ge c_4\|(\phi,\psi,\zeta)\|^2 + c_4\varepsilon^{-1}\|(\phi_y,\psi_y,\zeta_y)\|^2 \\ &\ge c_4\|(\Phi_y,\Psi_y,W_y)\|^2 + c_4\varepsilon^{-1}\|(\phi_y,\psi_y,\zeta_y)\|^2 - c_4\delta\varepsilon^4(1+\varepsilon^2\tau)^{-\frac{3}{2}},\end{aligned} \tag{2.74}$$

for some small positive constant c_4. Then, it follows from (2.73) and (2.74) that

$$\begin{cases}\|(\Phi_y,\Psi_y,W_y)\|^2 + \|(\phi,\psi,\zeta)\|^2 \le CC_0\sqrt{\delta}\varepsilon^2(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}, \\ \|(\phi_y,\psi_y,\zeta_y)\|^2 \le CC_0\sqrt{\delta}\varepsilon^3(1+\varepsilon^2\tau)^{-1+C_0\sqrt{\delta}}.\end{cases} \tag{2.75}$$

From (2.71) and (2.75), one has the decay rate for (Φ,Ψ,W),

$$\|(\Phi,\Psi,W)\|_{L^\infty} \le C\|(\Phi,\Psi,W)\|^{\frac{1}{2}}\|(\Phi_y,\Psi_y,W_y)\|^{\frac{1}{2}} \le CC_0\delta^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}(1+\varepsilon^2\tau)^{-\frac{1}{4}+\frac{1}{2}C_0\sqrt{\delta}}. \tag{2.76}$$

However, from (2.58), one observes that $\|(\phi_y,\psi_y,\zeta_y)\|^2$ should have better time-decay rate than the one in (2.75). In fact, multiplying (2.58) by $(1+\varepsilon^2\tau)^{\frac{3}{2}}$, one gets that

$$\begin{aligned}[(1+\varepsilon^2\tau)^{\frac{3}{2}}E_4]_\tau &\le C\delta\varepsilon^3\|(\Phi_y,\Psi_y,W_y)\|^2 + C\varepsilon(1+\varepsilon^2\tau)\|(\phi_y,\psi_y,\zeta_y)\|^2 \\ &\quad + C\delta\varepsilon^6(1+\varepsilon^2\tau)^{-\frac{3}{2}} + C(1+\varepsilon^2\tau)^{\frac{3}{2}}\|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}} \\ &\quad + C(1+\varepsilon^2\tau)^{\frac{3}{2}}\|(\phi,\psi)\|\|(\phi_y,\psi_y,\zeta_y)\|^3.\end{aligned} \tag{2.77}$$

By using (2.73) and (2.75), one has that

$$\begin{aligned}
&\int_0^\tau (1+\varepsilon^2 s)^{\frac{3}{2}} \|(\phi_y,\psi_y,\zeta_y)\|^{\frac{10}{3}} ds + \int_0^\tau (1+\varepsilon^2 s)^{\frac{3}{2}} \|(\phi,\psi)\| \|(\phi_y,\psi_y,\zeta_y)\|^3 ds \\
&\le \varepsilon^2 \int_0^\tau (1+\varepsilon^2 s)^{\frac{3}{2}} [(1+\varepsilon^2 s)^{-\frac{2}{3}+\frac{2}{3}C_0\sqrt{\delta}} + (1+\varepsilon^2 s)^{-1+C_0\sqrt{\delta}}] \|(\phi_y,\psi_y,\zeta_y)\|^2 ds \\
&\le \varepsilon^2 \int_0^\tau (1+\varepsilon^2 s) \|(\phi_y,\psi_y,\zeta_y)\|^2 ds \le CC_0\sqrt{\delta}\varepsilon^4 (1+\varepsilon^2\tau)^{C_0\sqrt{\delta}}, \qquad (2.78)
\end{aligned}$$

for $C_0\sqrt{\delta} \le \frac{1}{8}$. Thus, integrating (2.77) over $[0,\tau]$ and using (2.78), (2.73) and (2.75), one gets that

$$\|(\phi_y,\psi_y,\zeta_y)\|^2 \le CE_4 \le CC_0\sqrt{\delta}\varepsilon^3 (1+\varepsilon^2\tau)^{-\frac{3}{2}+C_0\sqrt{\delta}}.$$

By using Sobolev inequality, we obtain

$$\begin{aligned}
\|(\phi,\psi,\zeta)\|_{L^\infty} &\le \|(\phi,\psi,\zeta)\|^{\frac{1}{2}} \|(\phi_y,\psi_y,\zeta_y)\|^{\frac{1}{2}} \\
&\le C\delta^{\frac{1}{4}}\varepsilon^{\frac{5}{4}} (1+\varepsilon^2\tau)^{-\frac{5}{8}+\frac{1}{2}C_0\sqrt{\delta}} \le C\delta^{\frac{1}{4}}\varepsilon^{\frac{5}{4}} (1+\varepsilon^2\tau)^{-\frac{9}{16}},
\end{aligned}$$

for $C_0\sqrt{\delta} \le \frac{1}{8}$. Therefore the *a priori* assumption (2.10) is closed and the proof of Proposition 2.1 is completed. □

3. Ill-prepared data

3.1. Uniform estimates

For simplicity, throughout this subsection we omit the superscript ε of the variables and denote

$$a(\varepsilon p) = e^{-\varepsilon p}, \quad b(\theta) = e^{\theta}. \qquad (3.1)$$

For later use, we define that

$$\begin{cases}
\mathcal{Q}(t) := \sup_{0\le\tau\le t} \Big\{ \|(p,u)(\tau)\|_{\mathcal{H}^s} + \|(\varepsilon p,\varepsilon u)(\tau)\|_{\mathcal{H}^{s+1}} \\
\qquad\qquad + \|(\theta-\tilde{\theta})(\tau)\|_{L^2} + \|((\varepsilon\partial_t)\theta,\theta_x)(\tau)\|_{\mathcal{H}^s} \Big\}, \\
\mathcal{S}(t) := \|(p_x,u_x)(t)\|_{\mathcal{H}^s} + \|(\varepsilon u_x,\theta_x)(t)\|_{\mathcal{H}^{s+1}},
\end{cases} \qquad (3.2)$$

where we have used the notation $\|f(t)\|_{\mathcal{H}^s} := \sum_{|\alpha|\le s} \|\partial^\alpha f(t)\|_{L^2(\mathbb{R})}$ with $\partial^\alpha := (\varepsilon\partial_t)^{\alpha_0}\partial_x^{\alpha_1}$ and $\alpha = (\alpha_0,\alpha_1)$.

We assume the following a priori assumption

$$0 < \frac{1}{2}\underline{a} \le a(\varepsilon p) \le 2\bar{a}, \quad 0 < \frac{1}{2}\underline{b} \le b(\theta) \le 2\bar{b}, \text{ on } t \in [0, T]. \tag{3.3}$$

To prove Theorem 1.6, we need only to prove the following *a priori* estimates.

Proposition 3.1. *For any given integer $s \ge 4$ and $\varepsilon \in (0, 1]$, let $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ be the classical solution to the Cauchy problem (1.36) and (1.40). Under the a priori assumption (3.3), it holds that*

$$\mathcal{N}(t) \le C[1 + \Lambda(\mathcal{Q}(0))] + C(t^{\frac{1}{2}} + \varepsilon)\Lambda(\mathcal{N}(t)),$$

where $\mathcal{N}(t)$ is defined in (1.39), the constant $C > 0$ may depend on $\underline{a}, \bar{a}, \underline{b}$ and $\bar{b}$, and $\Lambda(\cdot)$ is a finite order polynomial.

Firstly, we have

Lemma 3.2. *For $s \ge 4$, it holds that*

$$\|\theta - \tilde{\theta}\|_{L^2}^2 + \sum_{|\alpha|=1}^{s} \|\partial^\alpha \theta\|_{L^2}^2 + \sum_{|\alpha|=0}^{s} \int_0^t \|\partial^\alpha \theta_x\|_{L^2}^2 d\tau \le C + \mathcal{Q}^2(0) + \int_0^t \Lambda(\mathcal{Q}(s))ds. \tag{3.4}$$

Proof. It follows from $(1.36)_3$ that

$$\begin{aligned}&(\theta - \tilde{\theta})_t + u(\theta - \tilde{\theta})_x + u_x - \kappa e^{-\varepsilon p}(e^\theta(\theta - \tilde{\theta})_x)_x \\ &= \tilde{\mu}\varepsilon^2 e^{-\varepsilon p}|u_x|^2 - u\tilde{\theta}_x + \kappa e^{-\varepsilon p}(e^\theta \tilde{\theta}_x)_x - \tilde{\theta}_t.\end{aligned} \tag{3.5}$$

Multiplying (3.5) by $\theta - \tilde{\theta}$ and integrating over $\mathbb{R}$, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |\theta - \tilde{\theta}|^2 dx + \kappa \int a(\varepsilon p)b(\theta)|(\theta - \tilde{\theta})_x|^2 dx + \int \kappa\varepsilon p_x e^{\theta - \varepsilon p}(\theta - \tilde{\theta})(\theta - \tilde{\theta})_x dx \\ &\le C\|u_x\|_{L^\infty}\|\theta - \tilde{\theta}\|_{L^2}^2 + C\|u_x\|_{L^2}\|\theta - \tilde{\theta}\|_{L^2} + C\varepsilon^2\|\theta - \tilde{\theta}\|_{L^\infty}\|u_x\|_{L^2}^2 \\ &\quad + \|\tilde{\theta}_x\|_{L^\infty}(\|u\|_{L^2} + \|(\theta - \tilde{\theta})_x\|_{L^2})\|\theta - \tilde{\theta}\|_{L^2} + C\|\theta - \tilde{\theta}\|_{L^2} \\ &\le C + \Lambda(\mathcal{Q}(t)).\end{aligned} \tag{3.6}$$

Integrating (3.6) with respect to time, one obtains that

$$\|(\theta - \tilde{\theta})(t)\|_{L^2}^2 + \int_0^t \|(\theta - \tilde{\theta})_x(\tau)\|_{L^2}^2 d\tau \le C + \|(\theta - \tilde{\theta})(0)\|_{L^2}^2 + \int_0^t \Lambda(\mathcal{Q}(\tau))d\tau. \tag{3.7}$$

Let $\theta_\alpha = \partial^\alpha \theta$, for $1 \le |\alpha| \le s$. Applying ∂^α to $(1.36)_3$, one obtains that

$$\begin{aligned}&\partial_t\theta_\alpha+u\partial_x\theta_\alpha+\partial^\alpha u_x-\kappa e^{-\varepsilon p}(e^\theta\partial_x\theta_\alpha)_x\\&=-[\partial^\alpha,u]\theta_x+\tilde{\mu}\varepsilon^2\partial^\alpha(e^{-\varepsilon p^\varepsilon}|u_x^\varepsilon|^2)+\kappa\Big\{\partial^\alpha(e^{-\varepsilon p}(e^\theta\theta_x)_x)-e^{-\varepsilon p}(e^\theta\partial_x\theta_\alpha)_x\Big\}.\end{aligned}\tag{3.8}$$

Multiplying (3.8) by θ_α and integrating the resulting equation over y, one has that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\theta_\alpha|^2dx+\int a(\varepsilon p)b(\theta)|\partial_x\theta_\alpha|^2dx+\frac{1}{2}\int u\partial_x(|\theta_\alpha|^2)dx+\int\theta_\alpha\partial^\alpha u_xdx\\&\le C\Big|\int a(\varepsilon p)b(\theta)\varepsilon p_x\theta_\alpha\partial_x\theta_\alpha dx\Big|+\|\theta_\alpha\|_{L^2}\|[\partial^\alpha,u]\theta_x\|_{L^2}+C\|\theta_\alpha\|_{L^2}\|\partial^\alpha(e^{-\varepsilon p^\varepsilon}|\varepsilon u_x^\varepsilon|^2)\|_{L^2}\\&\quad+C\Big|\int\theta_\alpha\Big[\partial^\alpha(e^{-\varepsilon p}(e^\theta\theta_x)_x)-e^{-\varepsilon p}(e^\theta\partial_x\theta_\alpha)_x\Big]dx\Big|.\end{aligned}\tag{3.9}$$

Integrating by parts and using the Cauchy inequality, we obtain that

$$\begin{aligned}&\Big|\int u\partial_x(|\theta_\alpha|^2)dx\Big|+\Big|\int\theta_\alpha\partial^\alpha u_xdx\Big|\le\|u_x\|_{L^\infty}\|\theta_\alpha\|_{L^2}^2+C\|\partial^\alpha u\|_{L^2}^2+C\|\partial_x\theta_\alpha\|_{L^2}^2\\&\le\|u\|_{H^2}\|\theta_\alpha\|_{L^2}^2+C\|\partial^\alpha u\|_{L^2}^2+C\|\partial_x\theta_\alpha\|_{L^2}^2\le\Lambda(\mathcal{Q}(t)).\end{aligned}\tag{3.10}$$

We shall estimate the RHS of (3.9). Firstly, we have that

$$\Big|\int a(\varepsilon p)b(\theta)\varepsilon p_x\theta_\alpha\partial_x\theta_\alpha dx\Big|\le C\|\varepsilon p\|_{H^2}^2\|\theta_\alpha\|_{L^2}^2+C\|\partial_x\theta_\alpha\|_{L^2}^2\le\Lambda(\mathcal{Q}(t)).\tag{3.11}$$

Notice that

$$[\partial^\alpha,u]\theta_x=\sum_{1\le|\beta|\le\alpha}C_{\alpha,\beta}\partial^\beta u\cdot\partial^{\alpha-\beta}\theta_x,\tag{3.12}$$

then one gets immediately that

$$\|[\partial^\alpha,u]\theta_x\|_{L^2}\le C\Big(\sum_{|\beta|\le s}\|\partial^\beta u\|_{L^2}\Big)\cdot\Big(\sum_{|\gamma|\le s-1}\|\partial^\gamma\theta_x\|_{L^2}\Big)\le C\Lambda(\mathcal{Q}(t)).\tag{3.13}$$

A directly calculation shows that

$$\begin{aligned}\|\partial^\alpha(e^{-\varepsilon p^\varepsilon}|\varepsilon u_x^\varepsilon|^2)\|_{L^2}&\le C\|\partial^\alpha(|\varepsilon u_x^\varepsilon|^2)\|_{L^2}+\Lambda(\|\varepsilon p\|_{\mathcal{H}^s})\cdot\|(\varepsilon u_x)^2\|_{\mathcal{H}^{s-1}}\\&\le C\|\varepsilon u_x^\varepsilon\|_{\mathcal{H}^s}^2+\Lambda(\|\varepsilon p\|_{\mathcal{H}^s})\cdot\|(\varepsilon u_x)^2\|_{\mathcal{H}^{s-1}}\le C\Lambda(\mathcal{Q}(t)).\end{aligned}\tag{3.14}$$

Noting that

$$\begin{aligned}&\partial^\alpha(e^{-\varepsilon p}(e^\theta\theta_x)_x)-e^{-\varepsilon p}(e^\theta\partial_x\theta_\alpha)_x\\&=\sum_{1\le|\beta|\le\alpha}C_{\alpha,\beta}\partial^\beta(e^{-\varepsilon p})\cdot\partial^{\alpha-\beta}(e^\theta\theta_x)_x+e^{-\varepsilon p}\Big(\sum_{1\le|\beta|\le\alpha}C_{\alpha,\beta}\partial(e^\theta)\cdot\partial^{\alpha-\beta}\theta_x\Big)_x,\end{aligned}$$

which yields immediately that

$$
\begin{aligned}
&\|\partial^\alpha(e^{-\varepsilon p}(e^\theta\theta_x)_x)-e^{-\varepsilon p}(e^\theta\partial_x\theta_\alpha)_x\|_{L^2}\\
&\le C\sum_{1\le|\beta|\le\alpha}\Big\{\|\partial^\beta(e^{-\varepsilon p})\cdot\partial^{\alpha-\beta}(e^\theta\theta_{xx}+e^\theta|\theta_x|^2)\|_{L^2}+\|\partial^\beta(e^\theta\theta_x)\cdot\partial^{\alpha-\beta}\theta_x\|\\
&\quad+\|\partial^\beta(e^\theta)\cdot\partial^{\alpha-\beta}\theta_{xx}\|\Big\}\\
&\le C\Lambda(\|\varepsilon p\|_{\mathcal{H}^s})\cdot\Big(\|\theta_{xx}\|_{\mathcal{H}^{s-1}}+\Lambda(\|\theta_x\|_{\mathcal{H}^s})\Big)+C\Lambda(\|\theta_x\|_{\mathcal{H}^s})\cdot[1+\|\theta_{xx}\|_{\mathcal{H}^{s-1}}]\\
&\le C\Lambda(\mathcal{Q}(t)).
\end{aligned}\tag{3.15}
$$

Substituting (3.10), (3.11), (3.13), (3.14) and (3.15) into (3.9), one obtains that

$$
\frac{1}{2}\frac{d}{dt}\int|\theta_\alpha|^2dx+\int a(\varepsilon p)b(\theta)|\partial_x\theta_\alpha|^2dx\le C\Lambda(\mathcal{Q}(t)).\tag{3.16}
$$

Integrating (3.16) with respect to time, one gets that

$$
\|\theta_\alpha\|_{L^2}^2+\int_0^t\|\partial_x\theta_\alpha(\tau)\|^2d\tau\le C\|\theta_\alpha(0)\|_{L^2}^2+C\int_0^t\Lambda(\mathcal{Q}(\tau))d\tau.\tag{3.17}
$$

Then, from (3.17) and (3.7), we obtain (3.4). Thus the proof of Lemma 3.2 is completed. □

Lemma 3.3. *For $s\ge4$, it holds that*

$$
\|(\varepsilon p,\varepsilon u)(t)\|_{\mathcal{H}^s}^2+\int_0^t\|\varepsilon u_x(\tau)\|_{\mathcal{H}^s}^2d\tau\le C\mathcal{Q}^2(0)+C\int_0^t\Lambda(\mathcal{Q}(\tau))\cdot[1+\mathcal{S}(\tau)]d\tau.\tag{3.18}
$$

Proof. For simplicity, we denote

$$
\breve{p}=\varepsilon p,\ \ \breve{u}=\varepsilon u,\quad\text{and}\quad(\breve{p}_\alpha,\breve{u}_\alpha)=\partial^\alpha(\breve{p},\breve{u}),\quad 0\le|\alpha|\le s.\tag{3.19}
$$

Then the functions $(\breve{p},\breve{u})$ satisfy

$$
\begin{cases}
\partial_t\breve{p}+u\breve{p}_x+(2u-a(\breve{p})b(\theta)\theta_x)_x=a(\breve{p})|\breve{u}_x|^2+a(\breve{p})b(\theta)\theta_x\cdot\breve{p}_x,\\
b(-\theta)[\partial_t\breve{u}+u\breve{u}_x]+p_x=\tilde{\mu}a(\breve{p})\breve{u}_{xx}.
\end{cases}\tag{3.20}
$$

Applying ∂^α to (3.20), one gets that

$$
\begin{cases}
\partial_t\breve{p}_\alpha+u\partial_x\breve{p}_\alpha=h_1+h_2+h_3+h_4,\\
b(-\theta)[\partial_t\breve{u}_\alpha+u\partial_x\breve{u}_\alpha]+\partial^\alpha p_x-\tilde{\mu}a(\breve{p})\partial_{xx}\breve{u}_\alpha=h_5+h_6+h_7,
\end{cases}\tag{3.21}
$$

where

$$\begin{cases} h_1 = -[\partial^\alpha, u]\partial_x \breve{p}, \quad h_2 = -\partial^\alpha \partial_x \Big(2u - a(\breve{p})b(\theta)\theta_x\Big), \\ h_3 = \partial^\alpha \Big(a(\breve{p})|\breve{u}_x|^2\Big), \quad h_4 = \partial^\alpha \Big(a(\breve{p})b(\theta)\theta_x \cdot \breve{p}_x\Big), \\ h_5 = -[\partial^\alpha, b(-\theta)]\partial_t \breve{u}, \quad h_6 = -[\partial^\alpha, b(-\theta)u]\partial_x \breve{u}, \quad h_7 = \tilde{\mu}[\partial^\alpha, a(\breve{p})]\breve{u}_{xx}. \end{cases} \tag{3.22}$$

Multiplying (3.21)$_1$ by $\breve{p}_\alpha$, one has that

$$\frac{1}{2}\frac{d}{dt}\int |\breve{p}_\alpha|^2 dx = \frac{1}{2}\int u_x |\breve{p}_\alpha|^2 dx + \int (h_1 + h_2 + h_3 + h_4)\breve{p}_\alpha dx. \tag{3.23}$$

It is straightforward to imply that

$$\Big|\int u_x |\breve{p}_\alpha|^2 dx\Big| \le \|u_x\|_{L^\infty} \|\breve{p}_\alpha\|_{L^2}^2 \le C\Lambda(\mathcal{Q}(t)). \tag{3.24}$$

A directly calculation shows that

$$\|(h_1, h_3, h_4)\|_{L^2} \le C\Lambda(\mathcal{Q}(t)),$$

which yields immediately that

$$\Big|\int (h_1 + h_3 + h_4)\breve{p}_\alpha dx\Big| \le C\Lambda(\mathcal{Q}(t)). \tag{3.25}$$

Noting that

$$\|h_2\|_{L^2} \le C[\|\partial^\alpha u_x\|_{L^2} + \|\partial^\alpha \partial_{xx}\theta\|_{L^2} + \Lambda(\mathcal{Q}(t))] \le C[\mathcal{S}(t) + \Lambda(\mathcal{Q}(t))], \tag{3.26}$$

we obtain that

$$\Big|\int h_2 \breve{p}_\alpha dx\Big| \le C\|\breve{p}_\alpha\|_{L^2} \cdot [\mathcal{S}(t) + \Lambda(\mathcal{Q}(t))]. \tag{3.27}$$

Substituting (3.24), (3.25) and (3.27) into (3.23), then integrating the resulting inequality with respect to time, then we get that

$$\|\breve{p}_\alpha\|^2 \le \|\breve{p}_\alpha(0)\|^2 + C\int_0^t \Lambda(\mathcal{Q}(\tau)) \cdot [1 + \mathcal{S}(\tau)] d\tau. \tag{3.28}$$

On the other hand, multiplying (3.21)$_2$ by $\breve{u}_\alpha$ and integrating the resulting equation, one obtains that

$$\frac{1}{2}\frac{d}{dt}\int b(-\theta)|\breve{u}_\alpha|^2 dx - \tilde{\mu}\int a(\breve{p})\breve{u}_\alpha \partial_{xx}\breve{u}_\alpha dx \le \int (h_5 + h_6 + h_7)\breve{u}_\alpha dx, \tag{3.29}$$

where we have used the facts

$$\left|\int \partial_t b(-\theta)|\breve{u}_\alpha|^2dx\right| + \left|\int b(-\theta)u(|\breve{u}_\alpha|^2)_x dx\right| \leq C\Lambda(\mathcal{Q}(t)), \tag{3.30}$$

and

$$\left|\int \partial^\alpha p_x \cdot \breve{u}_\alpha dx\right| \leq \|\partial^\alpha p\|_{L^2} \cdot \|\partial_x \breve{u}_\alpha\|_{L^2} \leq C\Lambda(\mathcal{Q}(t)). \tag{3.31}$$

For the second term on the left hand side of (3.29), it follows from integrating by parts that

$$\begin{aligned}-\tilde{\mu}\int a(\breve{p})\breve{u}_\alpha\partial_{xx}\breve{u}_\alpha dx &= \tilde{\mu}\int a(\breve{p})|\partial_x\breve{u}_\alpha|^2dx + \tilde{\mu}\int \partial_x a(\breve{p})\breve{u}_\alpha\partial_x\breve{u}_\alpha dx \\ &\geq \frac{3}{4}\tilde{\mu}\int a(\breve{p})|\partial_x\breve{u}_\alpha|^2dx - C\Lambda(\mathcal{Q}(t)). \end{aligned} \tag{3.32}$$

Now we estimate the RHS of (3.29). Noting

$$\|h_5\|_{L^2} = \|[\partial^\alpha, b(-\theta)]\partial_t\breve{u}\|_{L^2} = \|[\partial^\alpha, b(-\theta)](\varepsilon\partial_t)u\|_{L^2} \leq C\Lambda(\mathcal{Q}(t))\|u\|_{\mathcal{H}^s} \leq C\Lambda(\mathcal{Q}(t)),$$

and

$$\|h_6\|_{L^2} + \|h_7\|_{L^2} \leq C\Lambda(\mathcal{Q}(t)) + C\Lambda(\mathcal{Q}(t))\|\varepsilon u_x\|_{\mathcal{H}^s} \leq C\Lambda(\mathcal{Q}(t)),$$

which, together with Holder inequality, yield that

$$\left|\int (h_5 + h_6 + h_7)\breve{u}_\alpha dx\right| \leq \Lambda(\mathcal{Q}(t)). \tag{3.33}$$

Substituting (3.33)–(3.30) into (3.29) and integrating the resulting inequality with respect to time, one obtains that

$$\|\varepsilon u(t)\|^2_{\mathcal{H}^s} + \int_0^t \|\varepsilon u(\tau)\|^2_{\mathcal{H}^s}d\tau \leq C\|\varepsilon u(0)\|^2_{\mathcal{H}^s} + \int_0^t \Lambda(\mathcal{Q}(\tau))d\tau. \tag{3.34}$$

Combining (3.34) and (3.28), one proves (3.18). Thus the proof of Lemma 3.3 is completed. □

The following Lemma is devoted to the highest order derivative estimates $\|\partial^\alpha(\varepsilon p, \varepsilon u, \theta)(t)\|^2_{L^2}$, $|\alpha| = s + 1$.

Lemma 3.4. *For $s \geq 4$, it holds that*

$$\sum_{|\alpha|=s+1} \|\partial^\alpha(\varepsilon p, \varepsilon u, \theta)(t)\|_{L^2}^2 + \sum_{|\alpha|=s+1} \int_0^t \|\partial^\alpha(\varepsilon u_x, \theta_x)(\tau)\|_{L^2}^2 d\tau \\ \leq C\mathcal{Q}^2(0) + C\int_0^t [1+\mathcal{S}(\tau)]\Lambda(\mathcal{Q}(\tau))d\tau. \tag{3.35}$$

Proof. Similar to [2,25], we set

$$(\check{p}, \check{u}) = (\varepsilon p - \theta, \varepsilon u). \tag{3.36}$$

Then, from (1.36), it is easy to check that $(\check{p}, \check{u}, \theta)$ satisfy

$$\begin{cases} \partial_t \check{p} + u\check{p}_x + \frac{1}{\varepsilon}\check{u}_x = 0, \\ b(-\theta)[\partial_t \check{u} + u\check{u}_x] + \frac{1}{\varepsilon}(\check{p}_x + \theta_x) = \tilde{\mu} a(\varepsilon p)\check{u}_{xx}, \\ \partial_t \theta + u\theta_x + \frac{1}{\varepsilon}\check{u}_x = a(\varepsilon p)(b(\theta)\theta_x)_x + \tilde{\mu}\varepsilon^2 a(\varepsilon p)|u_x|^2. \end{cases} \tag{3.37}$$

Let α be a multi-index with $|\alpha| = s+1$ and denote

$$(\check{p}_\alpha, \check{u}_\alpha, \theta_\alpha) = \partial^\alpha(\check{p}, \check{u}, \theta). \tag{3.38}$$

Applying ∂^α to (3.37), one gets that

$$\begin{cases} \partial_t \check{p}_\alpha + u\partial_x \check{p}_\alpha + \frac{1}{\varepsilon}\partial_x \check{u}_\alpha = h_8, \\ b(-\theta)[\partial_t \check{u}_\alpha + u\partial_x \check{u}_\alpha] + \frac{1}{\varepsilon}(\partial_x \check{p}_\alpha + \partial_x \theta_\alpha) = \tilde{\mu} a(\varepsilon p)\partial_{xx}\check{u}_\alpha + h_9, \\ \partial_t \theta_\alpha + u\partial_x \theta_\alpha + \frac{1}{\varepsilon}\partial_x \check{u}_\alpha = a(\varepsilon p)(b(\theta)\partial_x \theta_\alpha)_x + h_{10}, \end{cases} \tag{3.39}$$

where

$$\begin{aligned} h_8 &= -[\partial^\alpha, u]\check{p}_x, \\ h_9 &= -[\partial^\alpha, b(-\theta)]\partial_t(\varepsilon u) - [\partial^\alpha, b(-\theta)u]\partial_x(\varepsilon u) + [\partial^\alpha, \tilde{\mu} a(\varepsilon p)]\partial_{xx}\check{u}, \\ h_{10} &= -[\partial^\alpha, u]\theta_x + \tilde{\mu}\partial^\alpha\Big(\varepsilon^2 a(\varepsilon p)|u_x|^2\Big) + \Big\{\partial^\alpha(a(\varepsilon p)(b(\theta)\theta_x)_x) - a(\varepsilon p)(b(\theta)\partial_x\theta_\alpha)_x\Big\}. \end{aligned}$$

Considering $\int \big[(3.39)_1 \times \check{p}_\alpha + (3.39)_2 \times \check{u}_\alpha + (3.39)_1 \times \theta_\alpha\big]dx$ and integrating by parts the resulting equation, one can obtain

$$\frac{1}{2}\frac{d}{dt}\int |\check{p}_\alpha|^2 + b(-\theta)|\check{u}_\alpha|^2 + |\theta_\alpha|^2 dx + \frac{3}{4}\int \tilde{\mu} a(\varepsilon p)|\partial_x \check{u}_\alpha|^2 + a(\varepsilon p)b(\theta)|\partial_x \theta_\alpha|^2 dx \\ \leq C\Lambda(\mathcal{Q}(t)) + \int (h_8 \check{p}_\alpha + h_9 \check{u}_\alpha + h_{10}\theta_\alpha)dx, \tag{3.40}$$

where we have used the following estimates

$$\Big|\int u\check{p}_\alpha \partial_x \check{p}_\alpha dx\Big| + \Big|\int \partial_t b(-\theta)\cdot|\check{u}_\alpha|^2 dx\Big| + \Big|\int b(\varepsilon p) u\check{u}_\alpha \partial_x \check{u}_\alpha dx\Big| \\ + \Big|\int u\theta_\alpha \partial_x \theta_\alpha dx\Big| \le \Lambda(\mathcal{Q}(t)), \tag{3.41}$$

$$\int a(\varepsilon p)\partial_{xx}\check{u}_\alpha \check{u}_\alpha dx = -\int a(\varepsilon p)|\partial_x \check{u}_\alpha|^2 dx - \int \partial_x a(\varepsilon p)\partial_x \check{u}_\alpha \check{u}_\alpha dx \\ \le -\frac{3}{4}\int a(\varepsilon p)|\partial_x \check{u}_\alpha|^2 dx + C\Lambda(\mathcal{Q}(t)), \tag{3.42}$$

$$\int a(\varepsilon p)(b(\theta)\partial_x\theta_\alpha)_x \theta_\alpha dx = -\int a(\varepsilon p) b(\theta)|\partial_x\theta_\alpha|^2 dx - \int \partial_x a(\varepsilon p)\cdot b(\theta)\theta_\alpha \partial_x \theta_\alpha dx \\ \le -\frac{3}{4}\int a(\varepsilon p) b(\theta)|\partial_x \check{u}_\alpha|^2 dx + C\Lambda(\mathcal{Q}(t)), \tag{3.43}$$

and the fact that

$$\int (\partial_x \check{u}_\alpha \check{p}_\alpha + \partial_x \check{p}_\alpha \check{u}_\alpha + \check{u}_\alpha \partial_x \theta_\alpha + \partial_x \check{u}_\alpha \theta_\alpha)\, dx = 0. \tag{3.44}$$

It remains to estimate the RHS of (3.40). Firstly, it follows from $(1.36)_2$ that

$$\varepsilon \partial_t u = -u\partial_x(\varepsilon u) - b(\theta)\partial_x p + \tilde{\mu} a(\varepsilon p) b(\theta)\partial_{xx}(\varepsilon u),$$

which yields immediately that

$$\|\varepsilon\partial_t u\|_{\mathcal{H}^s} \le C\Lambda(\mathcal{Q}(t))\Big\{1 + \|\partial_x(\varepsilon u)\|_{\mathcal{H}^s} + \|\partial_x p\|_{\mathcal{H}^s} + \|\partial_{xx}(\varepsilon u)\|_{\mathcal{H}^s}\Big\} \\ \le C\Lambda(\mathcal{Q}(t))\Big\{1 + \mathcal{S}(t)\Big\}. \tag{3.45}$$

Using (3.45), one gets that

$$\|h_8\|_{L^2} = \|[\partial^\alpha, u](\varepsilon p_x - \theta_x)\|_{L^2} \le C\|\varepsilon p_x - \theta_x\|_{\mathcal{H}^s}\|u\|_{\mathcal{H}^{s+1}} \\ \le \Lambda(\mathcal{Q}(t))\Big[\|\varepsilon\partial_t u\|_{\mathcal{H}^s} + \|u_x\|_{\mathcal{H}^s} + \|u\|_{\mathcal{H}^s}\Big] \\ \le C\Lambda(\mathcal{Q}(t))\Big\{1 + \mathcal{S}(t)\Big\},$$

which, together with Holder inequality, implies that

$$\Big|\int h_8 \check{p}_\alpha dx\Big| \le C\Lambda(\mathcal{Q}(t))\Big\{1 + \mathcal{S}(t)\Big\}. \tag{3.46}$$

Next, we shall estimate the terms involving h_9. It is straightforward to imply that

$$\|[\partial^\alpha, b(-\theta)u]\partial_x(\varepsilon u)\|_{L^2} + \|[\partial^\alpha, \tilde{\mu}a(\varepsilon p)]\partial_{xx}\check{u}\|_{L^2}$$
$$\leq C\Lambda(\mathcal{Q}(t))\Big[1 + \|\varepsilon u\|_{\mathcal{H}^{s+1}} + \|\varepsilon u_{xx}\|_{\mathcal{H}^s}\Big] \leq C\Lambda(\mathcal{Q}(t))\Big[1 + \mathcal{S}(t)\Big], \tag{3.47}$$

and

$$\|[\partial^\alpha, b(-\theta)]\partial_t(\varepsilon u)\|_{L^2} \leq C\Lambda(\mathcal{Q}(t))[1 + \|\varepsilon\partial_t u\|_{\mathcal{H}^s}] \leq C\Lambda(\mathcal{Q}(t))\Big[1 + \mathcal{S}(t)\Big], \tag{3.48}$$

where we have used (3.45) in the last inequality of (3.48). Then, it follows from (3.47) and (3.48) that

$$\Big|\int h_9 \check{u}_\alpha dx\Big| \leq C\Lambda(\mathcal{Q}(t))\Big[1 + \mathcal{S}(t)\Big]. \tag{3.49}$$

Finally, we estimate the terms involving h_{10}. Similarly, we have that

$$\|h_{10}\|_{L^2} \leq C\Lambda(\mathcal{Q}(t))\Big[1 + \mathcal{S}(t)\Big],$$

which yields that

$$\Big|\int h_{10}\theta_\alpha dx\Big| \leq C\Lambda(\mathcal{Q}(t))\Big[1 + \mathcal{S}(t)\Big]. \tag{3.50}$$

Substituting (3.50), (3.49) and (3.46) into (3.40) and integrating the resulting inequality with respect to time, one gets (3.35). Thus the proof of Lemma 3.4 is completed. □

To establish the estimates for $\|(p,u)\|_{\mathcal{H}^s}$, we first control the term $(\varepsilon\partial_t)^s(p,u)$ which plays key role in the estimates for $\|(p,u)\|_{\mathcal{H}^s}$. We start with a L^2-estimate for the following linearized equations around a given state $(\underline{p}, \underline{u}, \underline{\theta})$

$$\begin{cases} p_t^l + \underline{u}\cdot p_x^l + \frac{1}{\varepsilon}(2u^l - \kappa a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l)_x = \tilde{\mu}\varepsilon a(\varepsilon\underline{p})\underline{u}_x u_x^l + \kappa a(\varepsilon\underline{p})b(\underline{\theta})\underline{p}_x\theta_x^l + f_1, \\ b(-\underline{\theta})[u_t^l + \underline{u}\cdot u_x^l] + \frac{1}{\varepsilon}p_x^l = \tilde{\mu}a(\varepsilon\underline{p})u_{xx}^l + f_2, \\ \theta_t^l + \underline{u}\theta_x^l + u_x^l = \kappa a(\varepsilon\underline{p})(b(\underline{\theta})\theta_x^l)_x + \tilde{\mu}\varepsilon^2 a(\varepsilon\underline{p})\underline{u}_x u_x^l + f_3, \end{cases} \tag{3.51}$$

where we f_i, $i = 1, 2, 3$ are source terms.

Lemma 3.5. *Let (p^l, u^l, θ^l) be the solution of (3.51) and assume that*

$$0 < \frac{1}{2}\underline{a} \leq a(\varepsilon\underline{p}) \leq 2\bar{a}, \quad and \quad 0 < \frac{1}{2}\underline{b} \leq b(\underline{\theta}) \leq 2\bar{b}, \quad on\ t \in [0, T],$$

then it holds that, for $0 < t \leq T$,

$$\|(p^l, u^l)(t)\|_{L^2}^2 + \int_0^t \|u_x^l(\tau)\|_{L^2}^2 d\tau$$

$$\le C\|(p^l,u^l)(0)\|_{L^2}^2 + C\sup_{0\le\tau\le t}\|\theta_x^l(\tau)\|_{L^2}^2$$
$$+ C\Lambda(R_0)\Big\{\int_0^t \|(\varepsilon u_x^l,\theta_{xx}^l)\|_{L^2}^2 d\tau + \int_0^t \|(p^l,u^l,\theta_x^l)\|_{L^2}^2 d\tau + \int_0^t \|f_3\|_{L^2}^2 d\tau$$
$$+ \Big(\int_0^t \|(p^l,u^l,\theta_x^l)\|_{L^2}^2 d\tau\Big)^{\frac12}\cdot\Big(\int_0^t \|f_1,f_2\|_{L^2}^2 d\tau\Big)^{\frac12}\Big\}, \tag{3.52}$$

where $R_0 \doteq \sup_{\tau\in[0,T]}\{\|\partial_t\underline{\theta}(\tau)\|_{L^\infty} + \|(\underline{p},\underline{u},\underline{\theta})(\tau)\|_{W^{1,\infty}}\}$.

Proof. Following [25], we define

$$\mathfrak{u} := 2u^l - \kappa a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l. \tag{3.53}$$

Then, it is straightforward to check that p^l solves

$$p_t^l + \underline{u}\cdot p_x^l + \frac{1}{\varepsilon}\mathfrak{u}_x = \tilde{\mu}\varepsilon a(\varepsilon\underline{p})\underline{u}_x\mathfrak{u}_x + \frac{\tilde{\mu}}{2}\varepsilon a(\varepsilon\underline{p})\underline{u}_x(\kappa a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l)_x + \kappa a(\varepsilon\underline{p})b(\underline{\theta})\underline{p}_x\cdot\theta_x^l + f_1. \tag{3.54}$$

Consider $\frac12 a(\varepsilon\underline{p})\cdot\partial_x(3.51)_3$, one can get that

$$\begin{aligned}
&\frac12 b(-\underline{\theta})\Big[\partial_t(a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l) + \underline{u}\cdot(a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l)_x\Big]\\
&= \frac12 b(-\underline{\theta})\Big[\partial_t(a(\varepsilon\underline{p})b(\underline{\theta}))\theta_x^l + \underline{u}\cdot(a(\varepsilon\underline{p})b(\underline{\theta}))_x\theta_x^l\Big] - \frac{\kappa}{4}a(\varepsilon\underline{p})[a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l]_{xx}\\
&\quad + \frac{\tilde{\mu}}{4}\varepsilon^2 a(\varepsilon\underline{p})[a(\varepsilon\underline{p})\underline{u}_x\mathfrak{u}_x]_x + \frac{\tilde{\mu}}{4}\varepsilon^2 a(\varepsilon\underline{p})[a(\varepsilon\underline{p})\underline{u}_x\cdot(a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l)_x]_x - \frac12 a(\varepsilon\underline{p})\underline{u}_x\theta_x^l\\
&\quad + \frac12 a(\varepsilon\underline{p})[\kappa a(\varepsilon\underline{p})(b(\underline{\theta})\theta_x^l)_x]_x - \frac14 a(\varepsilon\underline{p})\mathfrak{u}_{xx} + \frac12 a(\varepsilon\underline{p})\cdot\partial_x f_3,
\end{aligned}$$

which, together with $(3.51)_2$, yields that

$$\begin{aligned}
&\frac12 b(-\underline{\theta})\Big[\partial_t\mathfrak{u} + \underline{u}\cdot\mathfrak{u}_x\Big] + \frac{1}{\varepsilon}p_x^l - \frac14 a(\varepsilon\underline{p})\mathfrak{u}_{xx} - \frac12\tilde{\mu}a(\varepsilon\underline{p})\mathfrak{u}_{xx}\\
&= -\frac12 b(-\underline{\theta})\Big[\partial_t(a(\varepsilon\underline{p})b(\underline{\theta}))\theta_x^l + \underline{u}\cdot(a(\varepsilon\underline{p})b(\underline{\theta}))_x\theta_x^l\Big] + \frac{\kappa}{4}a(\varepsilon\underline{p})[a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l]_{xx}\\
&\quad - \frac{\tilde{\mu}}{4}\varepsilon^2 a(\varepsilon\underline{p})[a(\varepsilon\underline{p})\underline{u}_x\mathfrak{u}_x]_x - \frac{\tilde{\mu}}{4}\varepsilon^2 a(\varepsilon\underline{p})[a(\varepsilon\underline{p})\underline{u}_x\cdot(a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l)_x]_x + \frac12 a(\varepsilon\underline{p})\underline{u}_x\theta_x^l\\
&\quad - \frac12 a(\varepsilon\underline{p})[\kappa a(\varepsilon\underline{p})(a(\underline{\theta})\theta_x^l)_x]_x + \frac{\tilde{\mu}}{2}a(\varepsilon\underline{p})\cdot[\kappa a(\varepsilon\underline{p})b(\underline{\theta})\theta_x^l]_{xx} + f_2 - \frac12 a(\varepsilon\underline{p})\cdot\partial_x f_3\\
&=: g + f_2 - \frac12 a(\varepsilon\underline{p})\cdot\partial_x f_3.
\end{aligned} \tag{3.55}$$

Multiplying (3.54) by p^l, (3.55) by $\mathfrak{u}$, adding them together and integrating the resulting equation, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\int \frac{1}{2}|p^l|^2 + \frac{1}{4}b(-\underline{\theta})|\mathfrak{u}|^2dx\right) + \int (\frac{1}{4} + \frac{1}{2}\tilde{\mu})a(\varepsilon\underline{p})|\mathfrak{u}_x|^2dx \\
&= \frac{1}{2}\int \underline{u}_x|p^l|^2dx + \frac{1}{4}\int [\partial_t b(-\underline{\theta}) + \partial_x(b(-\underline{\theta})\underline{u})]|\mathfrak{u}|^2dx + \int (\frac{1}{4} + \frac{1}{2}\tilde{\mu})\varepsilon a(\varepsilon\underline{p})\underline{p}_x\mathfrak{u}_x\mathfrak{u}dx \\
&\quad + \int \tilde{\mu}\varepsilon a(\varepsilon\underline{p})\underline{u}_x\mathfrak{u}_x p^l dx + \int \frac{\tilde{\mu}}{2}\varepsilon a(\varepsilon\underline{p})\underline{u}_x(\kappa a(\varepsilon\underline{p})b(\underline{\theta})\theta^l_x)_x p^l dx + \kappa\int a(\varepsilon\underline{p})b(\underline{\theta})\underline{p}_x\cdot\theta^l_x p^l dx \\
&\quad + \int \mathfrak{g}\mathfrak{u}dx + \int (f_1p^l + f_2\mathfrak{u})dx - \frac{1}{2}\int a(\varepsilon\underline{p})\partial_x f_3\cdot\mathfrak{u}dx \\
&\le \frac{1}{8}\int (\frac{1}{4} + \frac{1}{2}\tilde{\mu})a(\varepsilon\underline{p})|\mathfrak{u}_x|^2dx + C\Lambda(R_0)\|(p^l, u^l, \theta^l_x, \theta^l_{xx})\|^2 + \|f_3\|^2_{L^2} \\
&\quad + \int (f_1p^l + f_2\mathfrak{u})dx + \int \mathfrak{g}\mathfrak{u}dx. \qquad (3.56)
\end{aligned}$$

It is easy to note that

$$\left|\int \mathfrak{g}\mathfrak{u}dx\right| \le \frac{1}{8}\int (\frac{1}{4} + \frac{1}{2}\tilde{\mu})a(\varepsilon\underline{p})|\mathfrak{u}_x|^2dx + C\Lambda(R_0)\|(p^l, u^l, \theta^l_x, \theta^l_{xx})\|^2_{L^2}. \qquad (3.57)$$

Substituting (3.57) into (3.56) and integrating the resulting inequality with respect to time, one obtains (3.52). Therefore we complete the proof of Lemma 3.5. □

Next we shall use Lemma 3.5 to estimate $\|(\varepsilon\partial_t)^k(p,u)\|_{L^2}$ for $1 \le k \le s$.

Lemma 3.6. *For $s \ge 4$ and $0 \le k \le s$, it holds that*

$$\begin{aligned}
&\|(\varepsilon\partial_t)^k(p,u)(t)\|^2_{L^2} + \int_0^t \|(\varepsilon\partial_t)^k u_x(\tau)\|^2_{L^2}d\tau \\
&\le C\|(\varepsilon\partial_t)^k(p,u)(0)\|^2_{L^2} + C\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^2_{\mathcal{H}^k} + Ct^{\frac{1}{2}}\Lambda(\mathcal{N}(t)) \\
&\quad + C\Lambda(R)\int_0^t \|\theta_{xx}(\tau)\|^2_{\mathcal{H}^k}d\tau, \qquad (3.58)
\end{aligned}$$

where $R(t) = \sup_{0\le\tau\le t}\left\{\|\partial_t\theta(\tau)\|_{L^\infty} + \|(p,u,\theta)(\tau)\|_{W^{1,\infty}}\right\}$.

Proof. For simplicity, we set

$$(p_k, u_k, \theta_k) = (\varepsilon\partial_t)^k(p,u,\theta), \quad 0 \le k \le s. \qquad (3.59)$$

Applying $(\varepsilon\partial_t)^k$ to (1.36), one has that

$$\begin{cases} p_{kt} + u \cdot p_{kx} + \frac{1}{\varepsilon}(2u_k - \kappa a(\varepsilon p)b(\theta)\theta_{kx})_x = \tilde{\mu}\varepsilon a(\varepsilon p)u_x u_{kx} + \kappa a(\varepsilon p)b(\theta)p_x \cdot \theta_{kx} + f_{k1}, \\ b(-\theta)[u_{kt} + u \cdot u_{kx}] + \frac{1}{\varepsilon}p_{kx} = \tilde{\mu}a(\varepsilon p)u_{kxx} + f_{k2}, \\ \theta_{kt} + u\theta_{kx} + u_{kx} = \kappa a(\varepsilon p)(b(\theta)\theta_{kx})_x + \tilde{\mu}\varepsilon^2 a(\varepsilon p)u_x u_{kx} + f_{k3}, \end{cases} \tag{3.60}$$

where

$$\begin{cases} f_{k1} = -[(\varepsilon\partial_t)^k, u]p_x + \kappa\frac{1}{\varepsilon}\Big([(\varepsilon\partial_t)^k, a(\varepsilon p)b(\theta)]\theta_x\Big)_x + \varepsilon[(\varepsilon\partial_t)^k, \tilde{\mu}\varepsilon a(\varepsilon p)u_x]u_x \\ \qquad\qquad + [(\varepsilon\partial_t)^k, \kappa a(\varepsilon p)b(\theta)p_x]\theta_x, \\ f_{k2} = -[(\varepsilon\partial_t)^k, b(-\theta)]u_t - [(\varepsilon\partial_t)^k, b(-\theta)u]u_x + \tilde{\mu}[(\varepsilon\partial_t)^k, a(\varepsilon p)]u_{xx}, \\ f_{k3} = -[(\varepsilon\partial_t)^k, u]\theta_x + \tilde{\mu}\varepsilon[(\varepsilon\partial_t)^k, a(\varepsilon p)u_x]u_x + \kappa[(\varepsilon\partial_t)^k, a(\varepsilon p)](b(\theta)\theta_x)_x \\ \qquad\qquad + \kappa a(\varepsilon p)([(\varepsilon\partial_t)^k, b(\theta)]\theta_x)_x. \end{cases} \tag{3.61}$$

Applying Lemma 3.5 to equations (3.60), we obtain

$$\begin{aligned} &\|(\varepsilon\partial_t)^k(p,u)(t)\|_{L^2}^2 + \int_0^t \|(\varepsilon\partial_t)^k u_x\|_{L^2}^2 d\tau \\ &\le C\|(\varepsilon\partial_t)^k(p,u)(0)\|_{L^2}^2 + C\sup_{0\le\tau\le t}\|\theta_x(\tau)\|_{\mathcal{H}^k}^2 + C\Lambda(R)\Big\{\int_0^t \|(\varepsilon\partial_t)^k\theta_x x(\tau)\|_{L^2}^2 d\tau \\ &\quad + \int_0^t \|(p_k, u_k, (\theta_k)_x)\|_{L^2}^2 d\tau + \int_0^t \|f_{k3}\|_{L^2}^2 d\tau + t^{\frac{1}{2}}\Lambda(\mathcal{Q}(t))\Big(\int_0^t \|(f_{k1}, f_{k2})\|_{L^2}^2 d\tau\Big)^{\frac{1}{2}}\Big\}. \end{aligned} \tag{3.62}$$

It remains to estimate the terms involving (f_{k1}, f_{k2}, f_{k3}). We need only to estimate the case for $k \ge 1$ since $(f_{01}, f_{02}, f_{03}) = (0, 0, 0)$. For the singular term of f_{k1}, we first notice that

$$\Big([(\varepsilon\partial_t)^k, a(\varepsilon p)b(\theta)]\theta_x\Big)_x = [(\varepsilon\partial_t)^k, a(\varepsilon p)b(\theta)]\theta_{xx} + [(\varepsilon\partial_t)^k, \partial_x(a(\varepsilon p)b(\theta))]\theta_x, \tag{3.63}$$

then a careful calculation gives that

$$\begin{aligned} &\frac{1}{\varepsilon}\|[(\varepsilon\partial_t)^k, a(\varepsilon p)b(\theta)]\theta_{xx}\|_{L^2} = \|[(\varepsilon\partial_t)^{k-1}\partial_t, a(\varepsilon p)b(\theta)]\theta_{xx}\|_{L^2} \\ &\le \|[(\varepsilon\partial_t)^{k-1}, a(\varepsilon p)b(\theta)]\partial_t\theta_{xx}\|_{L^2} + \|[(\varepsilon\partial_t)^{k-1}, \partial_t(a(\varepsilon p)b(\theta))]\theta_{xx}\|_{L^2} \\ &\le C\Lambda(\mathcal{Q}(t))[1 + \|\partial_t\theta_x\|_{\mathcal{H}^{s-1}} + \|\partial_t\theta\|_{\mathcal{H}^{s-1}}], \end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\|[(\varepsilon\partial_t)^k,\partial_x(a(\varepsilon p)b(\theta))]\theta_x\|_{L^2}=\|[(\varepsilon\partial_t)^{k-1}\partial_t,\partial_x(a(\varepsilon p)b(\theta))]\theta_x\|_{L^2}\\
&\le\|[(\varepsilon\partial_t)^{k-1}\partial_t,\partial_{xt}(a(\varepsilon p)b(\theta))]\theta_x\|_{L^2}+\|[(\varepsilon\partial_t)^{k-1},\partial_x(a(\varepsilon p)b(\theta))]\theta_{xt}\|_{L^2}\\
&\le C\Lambda(\mathcal{Q}(t))[1+\|\partial_t\theta_x\|_{\mathcal{H}^{s-1}}+\|p_x\|_{\mathcal{H}^s}]. \qquad (3.65)
\end{aligned}$$

For the remain terms of f_{k1}, it is straightforward to obtain

$$\begin{aligned}
&\|[(\varepsilon\partial_t)^k,u]p_x\|_{L^2}+\|\varepsilon[(\varepsilon\partial_t)^k,\tilde{\mu}\varepsilon a(\varepsilon p)u_x]u_x\|_{L^2}+\|[(\varepsilon\partial_t)^k,\kappa a(\varepsilon p)b(\theta)p_x]\theta_x\|_{L^2}\\
&\le C\Lambda(\mathcal{Q}(t))[1+\|p_x\|_{\mathcal{H}^s}]. \qquad (3.66)
\end{aligned}$$

Combining (3.63)–(3.66), one gets that

$$\|f_{k1}\|_{L^2}\le C\Lambda(\mathcal{Q}(t))[1+\|\partial_t\theta_x\|_{\mathcal{H}^{s-1}}+\|p_x\|_{\mathcal{H}^s}]. \qquad (3.67)$$

For the estimate of $\|f_{k2}\|_{L^2}$, it is noted that

$$\begin{aligned}
[(\varepsilon\partial_t)^k,b(-\theta)]u_t&=\sum_{i=1}^k C_{k,i}(\varepsilon\partial_t)^i b(-\theta)\cdot(\varepsilon\partial_t)^{k-i}u_t\\
&=\sum_{i=1}^k C_{k,i}(\varepsilon\partial_t)^{i-1}\partial_t b(-\theta)\cdot(\varepsilon\partial_t)^{k-i+1}u,
\end{aligned}$$

which yields immediately that

$$\|[(\varepsilon\partial_t)^k,b(-\theta)]u_t\|_{L^2}\le C\Lambda(\mathcal{Q}(t))[1+\|\partial_t\theta\|_{\mathcal{H}^{k-1}}+\Lambda(\|\theta_t\|_{\mathcal{H}^{k-2}})], \qquad (3.68)$$

in which $\|\theta_t\|_{\mathcal{H}^{k-2}}$ does not appear when $k-2<0$. For the second and third terms of f_{k2}, it is straightforward to obtain that

$$\|[(\varepsilon\partial_t)^k,b(-\theta)u]u_x\|_{L^2}+\|\tilde{\mu}[(\varepsilon\partial_t)^k,a(\varepsilon p)]u_{xx}\|_{L^2}\le C\Lambda(\mathcal{Q}(t)). \qquad (3.69)$$

Then it follows from (3.68) and (3.69) that

$$\|f_{k2}\|_{L^2}\le C\Lambda(\mathcal{Q}(t))[1+\|\partial_t\theta\|_{\mathcal{H}^{k-1}}+\Lambda(\|\theta_t\|_{\mathcal{H}^{k-2}})]. \qquad (3.70)$$

Finally, a direct calculation yields that

$$\|f_{k3}\|_{L^2}\le C\Lambda(\mathcal{Q}(t)). \qquad (3.71)$$

Note that both (3.67) and (3.70) contain some norms of $\partial_t\theta$, which are not included in $\mathcal{Q}(t)$. To bound $\partial_t\theta$, we use $(1.36)_3$ to obtain

$$\|\theta_t\|_{\mathcal{H}^{s-1}}\le C\Lambda(\mathcal{Q}(t)),\quad\text{and}\quad\|\theta_{tx}\|_{\mathcal{H}^{s-1}}\le C\Lambda(\mathcal{Q}(t))\Big(1+\|u_x\|_{\mathcal{H}^s}+\|\theta_{xx}\|_{\mathcal{H}^s}\Big). \qquad (3.72)$$

Then, combining (3.72), (3.70) and (3.67), one gets that

$$\|(f_{k1}, f_{k2})\|_{L^2} \le C\Lambda(\mathcal{Q}(t))[1 + \|(p_x, u_x, \theta_{xx})\|_{\mathcal{H}^s}]. \tag{3.73}$$

Substituting (3.71) and (3.73) into (3.62), one proves (3.58). Therefore, the proof of Lemma 3.6 is completed. □

From Lemma 3.2, Lemma 3.4 and Lemma 3.6, we have the following corollary:

Corollary 3.7. *For $s \ge 4$ and $0 \le k \le s-1$, it holds that*

$$\|(\varepsilon\partial_t)^k (p,u)(t)\|_{L^2}^2 + \int_0^t \|(\varepsilon\partial_t)^k u_x(\tau)\|_{L^2}^2 d\tau \le C\Big(1 + \mathcal{Q}^2(0)\Big) + Ct^{\frac12}\Lambda(\mathcal{N}(t)), \tag{3.74}$$

and

$$\begin{aligned} &\|(\varepsilon\partial_t)^s (p,u)(t)\|_{L^2}^2 + \int_0^t \|(\varepsilon\partial_t)^s u_x(\tau)\|_{L^2}^2 d\tau \\ &\le C\Big(1 + \Lambda(\mathcal{Q}(0))\Big) + Ct^{\frac12}\Lambda(\mathcal{N}(t)) + C\Lambda(R(t)). \end{aligned} \tag{3.75}$$

where $R(t)$ is defined in Lemma 3.6.

We now use Corollary 3.7 to estimate $\|(p,u)\|_{\mathcal{H}^s}$. It follows from (1.36) that

$$\begin{cases} 2u_x = -\varepsilon\partial_t p - \varepsilon u p_x + (\kappa a(\varepsilon p) b(\theta)\theta_x)_x + \tilde{\mu} a(\varepsilon p)|\varepsilon u_x|^2 + \kappa a(\varepsilon p) b(\theta)\varepsilon p_x \cdot \theta_x, \\ p_x = -b(-\theta)\varepsilon\partial_t u - \varepsilon b(-\theta) u u_x + \tilde{\mu} a(\varepsilon p)\varepsilon u_{xx}. \end{cases} \tag{3.76}$$

Lemma 3.8. *It holds, for $s \ge 4$, that*

$$\|(p,u)\|_{\mathcal{H}^s} \le C\Big(1 + \Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac12} + \varepsilon)\Lambda(\mathcal{N}(t)). \tag{3.77}$$

Proof. It follows from $(3.76)_1$, Lemma 3.2–Lemma 3.4 and Corollary 3.7 that

$$\begin{aligned} \|u_x\|_{L^2} &\le C\Big\{\|(\varepsilon\partial_t)p\|_{L^2} + \varepsilon\|u\|_{H^1}\|p_x\|_{L^2} + \|\theta_{xx}\|_{L^2} + \Lambda(\|(\varepsilon p, \varepsilon u_x)\|_{\mathcal{H}^s} + \|\theta_x\|_{\mathcal{H}^{s-1}})\Big\} \\ &\le C\Big(1 + \Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac12} + \varepsilon)\Lambda(\mathcal{N}(t)). \end{aligned} \tag{3.78}$$

Similarly, one has, for $0 \le k \le s-2$, that

$$\begin{aligned} \|(\varepsilon\partial_t)^k u_x\|_{L^2} &\le C\Big\{\|(\varepsilon\partial_t)^{k+1}p\|_{L^2} + \varepsilon\Lambda(\mathcal{Q}) + \Lambda(\|(\varepsilon p, \varepsilon u)\|_{\mathcal{H}^s} + \|(\varepsilon\partial_t\theta, \theta_x, \theta_{xx})\|_{\mathcal{H}^{s-1}})\Big\} \\ &\le C\Big(1 + \Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac12} + \varepsilon)\Lambda(\mathcal{N}(t)), \end{aligned} \tag{3.79}$$

and

$$\begin{aligned}\|(\varepsilon\partial_t)^{s-1}u_x\|_{L^2} &\le C\Big\{\|(\varepsilon\partial_t)^s p\|_{L^2}+\varepsilon\Lambda(\mathcal{Q})+\Lambda(\|(\varepsilon p,\varepsilon u)\|_{\mathcal{H}^s}+\|(\varepsilon\partial_t\theta,\theta_x,\theta_{xx})\|_{\mathcal{H}^{s-1}})\Big\}\\ &\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t))+C\Lambda(R(t)).\end{aligned}\tag{3.80}$$

For p_x, it follows from (3.76)$_2$, Lemma 3.2–Lemma 3.4 and Corollary 3.7, for $0\le k\le s-2$, that

$$\begin{aligned}\|(\varepsilon\partial_t)^k p_x\|_{L^2} &\le C\Big\{\|(\varepsilon\partial_t)^{k+1}u\|_{L^2}+\varepsilon\Lambda(\mathcal{Q})+\Lambda(\|(\varepsilon p,\varepsilon u)\|_{\mathcal{H}^s}+\|(\varepsilon\partial_t\theta,\varepsilon u_{xx})\|_{\mathcal{H}^{s-1}})\Big\}\\ &\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)),\end{aligned}\tag{3.81}$$

and

$$\begin{aligned}\|(\varepsilon\partial_t)^{s-1}p_x\|_{L^2} &\le C\Big\{\|(\varepsilon\partial_t)^s u\|_{L^2}+\varepsilon\Lambda(\mathcal{Q})+\Lambda(\|(\varepsilon p,\varepsilon u)\|_{\mathcal{H}^s}+\|(\varepsilon\partial_t\theta,\varepsilon u_{xx})\|_{\mathcal{H}^{s-1}})\Big\}\\ &\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t))+C\Lambda(R(t)).\end{aligned}\tag{3.82}$$

Using the system (3.76) and (3.79), (3.81), one obtains, for $0\le k\le s-3$ that

$$\|(\varepsilon\partial_t)^k(p_{xx},u_{xx})\|_{L^2}\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).\tag{3.83}$$

Similarly, one can get that

$$\|(p,u)\|_{\mathcal{H}^{s-1}}\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).\tag{3.84}$$

Then it follows from (1.36)$_3$, (3.4) and (3.84) that

$$R(t)\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).\tag{3.85}$$

Combining (3.75) (3.76), (3.80), (3.82), (3.84), (3.85), and using the same argument as in (3.84), one can get that

$$\|(p,u)\|_{\mathcal{H}^{s}}\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).\tag{3.86}$$

Therefore the proof of Lemma 3.8 is completed. □

Lemma 3.9. *It holds, for $s\ge 4$, that*

$$\int_0^t\|(p_x,u_x)(\tau)\|^2_{\mathcal{H}^s}d\tau\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big)+C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).\tag{3.87}$$

Proof. Firstly, it follows from (3.75) and (3.85) that

$$\int_0^t \|(\varepsilon\partial_t)^s u_x(\tau)\|_{L^2}^2 d\tau \le C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)). \tag{3.88}$$

Using (3.88) and $(3.76)_1$, one can obtain that

$$\begin{aligned}
&\int_0^t \|(\varepsilon\partial_t)^{s+1} p(\tau)\|_{L^2}^2 d\tau \\
&\le C\int_0^t \|(\varepsilon\partial_t)^s u_x(\tau)\|_{L^2}^2 d\tau + C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)) \\
&\le C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).
\end{aligned} \tag{3.89}$$

On the other hand, it follows from $(3.76)_2$ that

$$(\varepsilon\partial_t)^s p_x = -\varepsilon b(-\theta)(\varepsilon\partial_t)^s u_t + (\varepsilon\partial_t)^s\Big\{-\varepsilon b(-\theta)uu_x + \tilde{\mu}a(\varepsilon p)\varepsilon u_{xx}\Big\}. \tag{3.90}$$

Multiplying (3.90) by $(\varepsilon\partial_t)^s p_x$ and integrating the resulting equation yield that

$$\begin{aligned}
\|(\varepsilon\partial_t)^s p_x\|_{L^2}^2 &= -\varepsilon\int b(-\theta)(\varepsilon\partial_t)^s u_t\cdot(\varepsilon\partial_t)^s p_x dx \\
&\quad + \int (\varepsilon\partial_t)^s p_x\cdot(\varepsilon\partial_t)^s\Big\{-\varepsilon b(-\theta)uu_x + \tilde{\mu}a(\varepsilon p)\varepsilon u_{xx}\Big\}dx \\
&\le -\frac{d}{dt}\int b(-\theta)(\varepsilon\partial_t)^s u\cdot(\varepsilon\partial_t)^s(\varepsilon p_x)dx + \int b(-\theta)(\varepsilon\partial_t)^s u\cdot(\varepsilon\partial_t)^{s+1}p_x dx \\
&\quad + \int \partial_t b(-\theta)(\varepsilon\partial_t)^s u\cdot(\varepsilon\partial_t)^s(\varepsilon p)_x dx + \frac{1}{8}\|(\varepsilon\partial_t)^s p_x\|_{L^2}^2 \\
&\quad + \int \Big|(\varepsilon\partial_t)^s\Big\{-\varepsilon b(-\theta)uu_x + \tilde{\mu}a(\varepsilon p)\varepsilon u_{xx}\Big\}\Big|^2 dx.
\end{aligned} \tag{3.91}$$

Integrating (3.91) with respect to time, using (3.88), (3.89), Lemma 3.2–Lemma 3.4 and Lemma 3.8, one immediately gets that

$$\int_0^t \|(\varepsilon\partial_t)^s p_x(\tau)\|_{L^2}^2 d\tau \le C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)). \tag{3.92}$$

Then, from (3.76), (3.89) and (3.92), one obtains that

$$\int_0^t \|(\varepsilon\partial_t)^{s-1}(p_{xx}, u_{xx})(\tau)\|_{L^2}^2 d\tau$$
$$\leq C\int_0^t \|(\varepsilon\partial_t)^{s}(p_{x}, u_{x})(\tau)\|_{L^2}^2 d\tau + C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t))$$
$$\leq C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)). \tag{3.93}$$

In the same way, one can prove that

$$\int_0^t \|(p_x, u_x)(\tau)\|_{\mathcal{H}^s}^2 d\tau \leq C\Big(1+\Lambda(\mathcal{Q}(0))\Big) + C(t^{\frac{1}{2}}+\varepsilon)\Lambda(\mathcal{N}(t)).$$

Therefore the proof of Lemma 3.9 is completed. □

Proof of Proposition 3.1. Proposition 3.1 follows immediately from Lemma 3.2–Lemma 3.4, Lemma 3.8 and Lemma 3.9. □

Proof of Theorem 1.6. Using Proposition 3.1, one can prove Theorem 1.6 by combining the local existence theorem and the bootstrap arguments, and thus close the a priori assumption (3.3). The details are omitted here for simplicity of presentation. □

3.2. Low Mach limit

In this subsection, we will prove Theorem 1.9 with a modified compactness argument, which was introduced by Métivier and Schochet in [35], see also the extensions in [1,2,31].

Proof of Theorem 1.9. Firstly, it follows from Theorem 1.6 that

$$\sup_{\tau\in[0,T_0]} \|(p^\varepsilon, u^\varepsilon)(\tau)\|_{H^s} + \sup_{\tau\in[0,T_0]} \|(\theta^\varepsilon - \tilde\theta)(\tau)\|_{H^{s+1}} < \infty. \tag{3.94}$$

Then extracting a subsequence, it holds that

$$(p^\varepsilon, u^\varepsilon) \rightharpoonup (\overline{p}, \overline{u}) \ \text{ as } \ \varepsilon\to 0 \ \text{ weak-}* \ \text{ in } \ L^\infty(0, T_0; H^s(\mathbb{R})),$$
$$\theta^\varepsilon - \tilde\theta \rightharpoonup \bar\theta - \tilde\theta \ \text{ as } \ \varepsilon\to 0 \ \text{ weak-}* \ \text{ in } \ L^\infty(0, T_0; H^{s+1}(\mathbb{R})).$$

It follows from the equation of θ^ε and (3.94) that

$$\theta_t^\varepsilon \in L^\infty(0, T_0; H^{s-2}(\mathbb{R})),$$

which, together with Aubin–Lions lemma, yields that the functions θ^ε converge (possibly after extracting a subsequence) to $\bar\theta$ strongly in $C([0, T_0]; H^{s'+1}_{loc}(\mathbb{R}))$ for all $s' < s$.

To obtain the limiting system (1.37), we need to show the strong convergence of $(p^\varepsilon, u^\varepsilon)$ in $L^2(0,T_0;H^{s'}_{loc}(\mathbb{R}))$ for $s' < s$. To this end, we will show p^ε and $(2u^\varepsilon - \kappa e^{\theta^\varepsilon-\varepsilon p^\varepsilon}\theta^\varepsilon_x)_x$ converge strongly to 0 as $\varepsilon \to 0$. In fact we rewrite $(1.36)_1$ and $(1.36)_2$ as,

$$\varepsilon p^\varepsilon_t + (2u^\varepsilon - \kappa e^{\theta^\varepsilon-\varepsilon p^\varepsilon}\theta^\varepsilon_x)_x = \varepsilon f^\varepsilon, \tag{3.95}$$

and

$$\varepsilon e^{-\theta^\varepsilon} u^\varepsilon_t + p^\varepsilon_x = \varepsilon g^\varepsilon. \tag{3.96}$$

It follows from (1.42) that f^ε and g^ε are uniformly bounded in $C([0,T_0];H^{s-1}(\mathbb{R}))$. Noticing p^ε are uniformly bounded in $L^\infty(0,T_0,L^\infty(\mathbb{R}))$, passing the weak limit in (3.95) and (3.96) leads to $(\overline{p})_x = 0$ and $(2\overline{u} - \kappa e^{\bar\theta}\bar\theta_x)_x = 0$ in the distribution sense.

On the other hand, by taking the limit of $(1.36)_3$, we can get that $\bar\theta$ satisfies

$$\bar\theta_t = \frac{\kappa e^{\bar\theta}}{2}\bar\theta_{xx}, \tag{3.97}$$

with the initial data,

$$\bar\theta(x,0) = \theta_{in}(x). \tag{3.98}$$

From the maximum principle and energy method, one can show the existence and uniqueness of smooth solution of (3.97), (3.98). To get the spatial decay of $\bar\theta$, as in [2], we define:

$$H = x^{1+\sigma}(\bar\theta - \theta_+),$$

which satisfies

$$H_t = \frac{\kappa e^{\bar\theta}}{2}H_{xx} - \frac{\kappa(1+\sigma)e^{\bar\theta}}{x}H_x + \frac{\kappa(1+\sigma)(2+\delta)e^{\bar\theta}}{x^2}H.$$

By the energy estimates and (1.42), we can obtain

$$\|H\|_{L^\infty(0,T_0;H^1[1,\infty))} \le C[\|H(0)\|_{H^1[1,\infty)} + \|\bar\theta - \tilde\theta\|_{H^2} + 1] \le C[1+\Lambda(C_0)]$$

which, together with Sobolev embedding, yields that

$$|\bar\theta(x,t) - \theta_+| \le Cx^{-1-\sigma}, \text{ as } x \in [1,+\infty). \tag{3.99}$$

To obtain the strong convergence of u^ε and p^ε, we need the following Proposition 3.10 which will be proved in the end of this section.

Proposition 3.10. *Let (1.42) and (3.99) hold, then p^ε and $(2u^\varepsilon - \kappa e^{\theta^\varepsilon-\varepsilon p^\varepsilon}\theta^\varepsilon_x)_x$ converge to 0 strongly in $L^2(0,T_0;H^{s'}_{loc}(\mathbb{R}))$ and $L^2(0,T_0;H^{s'-1}_{loc}(\mathbb{R}))$ for $s' < s$, respectively.*

If Proposition 3.10 holds, passing the limit in the equations (1.36) for $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$, one proves that the limit $(0, \overline{u}, \overline{\theta})$ solves (1.37) in the sense of distribution.

On the other hand, following the arguments in [35], one can obtain that $(\bar{u}, \bar{\theta})$ satisfies the initial condition

$$(\bar{u}, \bar{\theta})|_{t=0} = (w_{in}, \theta_{in}) \tag{3.100}$$

where w_{in} is determined by $w_{in} = \frac{1}{2}\kappa e^{\theta_{in}}(\theta_{in})_x$. Moreover one can get the uniqueness of solutions to the limit system (1.37) with initial data (3.100) by the energy method and then the above conclusions hold for the whole sequence $(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$. The proof of Theorem 1.9 is completed. □

Proof of Proposition 3.10. Applying $\varepsilon\partial_t$ to (3.95) and ∂_x to $e^{\theta^\varepsilon} \times$ (3.96), we obtain that

$$\varepsilon^2 \frac{1}{2} p^\varepsilon_{tt} - (e^{\theta^\varepsilon} p^\varepsilon_x)_x = \varepsilon F^\varepsilon(p^\varepsilon, u^\varepsilon, \theta^\varepsilon), \tag{3.101}$$

where $F^\varepsilon(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ is a smooth function. From (1.42), $\varepsilon F^\varepsilon(p^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ converges to 0 strongly in $L^2(0, T_0; L^2(\mathbb{R}))$ as $\varepsilon \to 0$. We now recall the convergence lemma due to Métivier and Schochet [35] in $\mathbb{R}^d$ with $d = 1, 2, 3$.

Lemma 3.11. *Let $T > 0$, v^ε be a bounded sequence in $C([0,T], H^2(\mathbb{R}^d))$ and $\varepsilon\partial_t v^\varepsilon$ are bounded in $L^2(0,T; L^2(\mathbb{R}^d))$ satisfying*

$$\varepsilon^2 \partial_t(a^\varepsilon \partial_t v^\varepsilon) - \nabla \cdot (b^\varepsilon \nabla v^\varepsilon) = c^\varepsilon, \tag{3.102}$$

where c^ε converges to 0 strongly in $L^2(0,T; L^2(\mathbb{R}^d))$. Assume further that, for some $k > 1 + d/2$, the coefficients $(a^\varepsilon, b^\varepsilon)$ are positive and uniformly bounded in $C([0,T]; H^k_{loc}(\mathbb{R}^d))$ and converge in $C([0,T]; H^k_{loc}(\mathbb{R}^d))$ to (a, b) satisfying

$$\text{for all } \tau \in \mathbb{R}, \text{ the kernel of } a\tau^2 + \nabla \cdot (b\nabla) \text{ in } L^2(\mathbb{R}^d) \text{ is reduced to } \{0\}. \tag{3.103}$$

Then the sequence v^ε converges to 0 strongly in $L^2(0,T; L^2_{loc}(\mathbb{R}^d))$.

We introduce the following condition to verify (3.103).

Condition A. *The functions (a, b) are positive bounded and satisfy*

$$|a(x,t) - a_+| \le C_a r(x), \quad |b(x,t) - b_+| \le C_b r(x), \text{ as } x \in [1, +\infty),$$

where $r(x) \in L^1([1, +\infty))$ is a non-negative function, and a_+, b_+, C_a, C_b are some positive constants.

It is obvious that if $v \in L^2(\mathbb{R})$ satisfies

$$a\tau^2 v + \partial_x(b\partial_x v) = 0, \tag{3.104}$$

then $v \in H^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} b(\partial_x v)^2 dx = \tau^2 \int_{\mathbb{R}} av^2 dx,$$

which implies $v \equiv 0$ when $\tau = 0$. When $\tau \neq 0$, we assume $\tau = 1$ without loss of generality. Let $w := b\partial_x v$, then (3.104) becomes

$$\frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & h \\ -a & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \tag{3.105}$$

where $h := 1/b$ has similar properties of b. From $v \in H^1(\mathbb{R})$, $v, w \to 0$ as $|x| \to \infty$.

For further analysis, we need the following Lemma 3.12 which can be verified directly by the energy method. The details are omitted here.

Lemma 3.12. *Let $A(x)$ be a smooth bounded function on $x \in \mathbb{R}^+$. Assume $\frac{d}{dx}U = A(x)U$, $U(0) = 0$ with $U \in \mathbb{C}^d$. Then $U \equiv 0$.*

We shall first prove $(v, w)^T(x) = 0$ for $x \in [1, +\infty)$. It is noted that $(a, h) \to (a_+, h_+)$ as $x \to +\infty$. Then we rewrite (3.105) as

$$\begin{aligned} \frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 & -h_+ \\ a_+ & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} 0 & h - h_+ \\ -(a - a_+) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \\ &=: \begin{pmatrix} 0 & \tilde{h} \\ -\tilde{a} & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}. \end{aligned} \tag{3.106}$$

A direct calculation shows that $B = Q^{-1}\Lambda Q$ with

$$B = \begin{pmatrix} 0 & -h_+ \\ a_+ & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \sqrt{a_+h_+}i & 0 \\ 0 & -\sqrt{a_+h_+}i \end{pmatrix} \text{ and } Q = \begin{pmatrix} \sqrt{a_+} & \sqrt{h_+}i \\ \sqrt{a_+} & -\sqrt{h_+}i \end{pmatrix}. \tag{3.107}$$

It follows from (3.106) and (3.107) that

$$\begin{aligned} &\frac{d}{dx}\begin{pmatrix} \sqrt{a_+}v + \sqrt{h_+}iw \\ \sqrt{a_+}v - \sqrt{h_+}iw \end{pmatrix} + \begin{pmatrix} \sqrt{a_+h_+}i & 0 \\ 0 & -\sqrt{a_+h_+}i \end{pmatrix}\begin{pmatrix} \sqrt{a_+}v + \sqrt{h_+}wi \\ \sqrt{a_+}v - \sqrt{h_+}wi \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{a_+} & \sqrt{h_+}i \\ \sqrt{a_+} & -\sqrt{h_+}i \end{pmatrix}\begin{pmatrix} 0 & \tilde{h} \\ -\tilde{a} & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \end{aligned} \tag{3.108}$$

which yields immediately that

$$\frac{d}{dx}\begin{pmatrix} e^{\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v+\sqrt{h_+}wi\right) \\ e^{-\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v-\sqrt{h_+}wi\right) \end{pmatrix}$$
$$=\begin{pmatrix} \sqrt{a_+}e^{\sqrt{a_+h_+}ix} & \sqrt{h_+}ie^{\sqrt{a_+h_+}ix} \\ \sqrt{a_+}e^{-\sqrt{a_+h_+}ix} & -\sqrt{h_+}ie^{-\sqrt{a_+h_+}ix} \end{pmatrix}\begin{pmatrix} 0 & \tilde{h} \\ -\tilde{a} & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}.$$

Then, one has that

$$\frac{d}{dx}\begin{pmatrix} e^{\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v+\sqrt{h_+}wi\right) \\ e^{-\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v-\sqrt{h_+}wi\right) \end{pmatrix}=\tilde{B}\begin{pmatrix} e^{\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v+\sqrt{h_+}wi\right) \\ e^{-\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v-\sqrt{h_+}wi\right) \end{pmatrix},$$

where

$$\tilde{B}:=\begin{pmatrix} \sqrt{a_+}e^{\sqrt{a_+h_+}ix} & \sqrt{h_+}ie^{\sqrt{a_+h_+}ix} \\ \sqrt{a_+}e^{-\sqrt{a_+h_+}ix} & -\sqrt{h_+}ie^{-\sqrt{a_+h_+}ix}i \end{pmatrix}\begin{pmatrix} 0 & \tilde{h} \\ -\tilde{a} & 0 \end{pmatrix}$$
$$\times\begin{pmatrix} \sqrt{a_+}e^{\sqrt{a_+h_+}ix} & \sqrt{h_+}ie^{\sqrt{a_+h_+}ix} \\ \sqrt{a_+}e^{-\sqrt{a_+h_+}ix} & -\sqrt{h_+}ie^{-\sqrt{a_+h_+}ix} \end{pmatrix}^{-1}$$
$$=\frac{i}{2\sqrt{a_+h_+}}\begin{pmatrix} -h_+\tilde{a}-a_+\tilde{h} & (a_+\tilde{h}-h_+\tilde{a})e^{2\sqrt{a_+h_+}ix} \\ (h_+\tilde{a}-a_+\tilde{h})e^{-2\sqrt{a_+h_+}ix} & h_+\tilde{a}+a_+\tilde{h} \end{pmatrix}.$$

We introduce a new coordinate $y=\int_x^{+\infty}r(z)dz$ satisfying

$$\frac{d}{dx}=\frac{dy}{dx}\frac{d}{dy}=-r(x)\frac{d}{dy}.$$

Define

$$A:=\frac{-1}{r(x)}\tilde{B}\quad\text{and}\quad U(y):=\begin{pmatrix} e^{\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v+\sqrt{h_+}wi\right) \\ e^{-\sqrt{a_+h_+}ix}\left(\sqrt{a_+}v-\sqrt{h_+}wi\right) \end{pmatrix}.$$

From **Condition A**, we can check that A is a smooth bounded function. The initial data $U(0)=0$ follows from the fact $\lim_{x\to\infty}(v,w)(x)=0$. Therefore, it follows from Lemma 3.12 that $(v,w)(x)\equiv 0$ for $x\in[1,+\infty)$. Going back to the original ODE system (3.105) and employing Lemma 3.12 again, one can prove $v\equiv 0$ from which (3.103) holds.

Now, we return to the convergence of p^ε. By the strong convergence of θ^ε and (3.99), one can prove that the coefficients in (3.101) satisfy **Condition A**. It follows from Lemma 3.11 that $p^\varepsilon\to 0$ strongly in $L^2(0,T_0;L^2_{loc}(\mathbb{R}))$ as $\varepsilon\to 0$. On the other hand, the uniform boundedness (1.42) and the interpolation theorem yield immediately that

$$p^{\varepsilon} \to 0 \text{ strongly in } L^2(0,T_0;H^{s'}_{loc}(\mathbb{R})), \text{ for } s' < s.$$

In the same way, we can prove $(2u^{\varepsilon} - \kappa e^{\theta^{\varepsilon}-\varepsilon p^{\varepsilon}}\theta^{\varepsilon}_x)_x \to 0$ as $\varepsilon \to 0$. Therefore the proof of Proposition 3.10 is completed. □

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